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BIOMETRIKA

THE CRANIAL COORDINATOGRAPH, THE STANDARD PLANES OF THE SKULL, AND THE VALUE OF CARTESIAN GEOMETRY TO THE CRANIOLOGIST, WITH SOME ILLUSTRATIONS OF THE USES OF THE NEW METHOD*.

BY KARL PEARSON.

1. *Introductory.*

Many years ago when I had more leisure to give to craniometry than I have had recently, I became convinced of the unsatisfactory nature of the "standard" planes of the skull, and the need for a careful revision of the whole subject. In particular, when one had realised the absolute asymmetry of the skull in all parts and in all directions, it seemed irrational to suppose that the auricular axis was likely to lie in a "horizontal" plane. Assuming for a moment that there is such an entity as a "median sagittal" plane, or plane with regard to which the absolutely symmetrical skull should have mirror symmetry, then in the natural skull the ears would be shifted right and left, forward and backward, up and down with regard to this mirror plane of symmetry, and accordingly the auricular axis would make an angle with the mirror plane of symmetry. Further, all the fundamental points of the skull which should lie in the mirror plane, i.e. the "mid-sagittal" points, would in an actual skull be dispersed some to the right and some to the left of it, and the fiction of a median sagittal plane as a true *standard vertical plane* of the skull, seemed to vanish with the asymmetry of the skull.

Convinced that, with an asymmetrical system like the skull, the main object must still be to start from a fitly chosen median sagittal section as the standard median vertical plane, and that the standard horizontal plane must be perpendicular to this, I could only look upon the Frankfurt Horizontal Plane and the Transverse Vertical Plane, both passing through the auricular axis, and the Median Sagittal Plane (as determined by any three points†) as very temporary and inadequate expedients to obtain three mutually perpendicular standard cranial planes. These customary planes are not mutually rectangular. By construction the Frankfurt Horizontal Plane and the usual Transverse Vertical Plane are at right angles. Hence the third plane ought to be at right angles to both these planes, that is to say at right angles to the auricular axis. It is perfectly easy to draw the curve of intersection of such a

* The material points of this paper were given as a lecture before the Oxford University Anthropological Society on May 25, 1933.

† These three points are assumed to be in the "mirror plane of symmetry," for example Nasion, Bregma and Lambda as used in the Biometric Laboratory, or Nasion, Inion and Basion as suggested by Martin. See p. 227 below.

plane with the external surface of the skull, if we have an instrument for setting the skull with its auricular axis perpendicular to the plane of the craniometric table. That is possible by aid of the cranial coordinatograph. It only remains to settle through what point this plane perpendicular to the auricular axis shall be taken. If the skull had complete mirror symmetry then this plane should bisect the auricular axis, i.e. pass through the point—the Mid-porion—midway between the Right and Left Poria. In the random selection of crania I have used to illustrate this paper, this Mid-porion plane deviates so widely from any supposed median sagittal section, that no one would think of using it. By aid of my coordinatograph, Dr Morant kindly drew for me on a Hindu skull the Frankfurt Horizontal Plane*, the usual Transverse Vertical Plane, as determined by placing the skull on a Ranke craniophor, and the Mid-porion perpendicular plane. Plates I—III indicate the absurd results thus reached. The absurdity lies in the fact that in no ordinary skull is the auricular axis perpendicular to any reasonable Median Sagittal Plane, and the sooner we, as craniologists, realise this the better. The auricular axis makes an angle differing from a right angle with the Median Sagittal Plane and has no real claim to be selected as a "horizontal line." A very brief experience will convince an attentive observer that when the subject is holding his head "straight" the two ears are not usually on the same level, to say nothing of their equality in distance from any mid-line of the face†.

I have, perhaps, said enough to convince the reader that the fundamental crux of the determination of the standard planes of the skull lies in the discovery of an adequately satisfactory "median sagittal section," i.e. a standard vertical sagittal section or an approximate mirror plane. This must precede the determination of a standard horizontal plane, if only for the strong reason that with a truly symmetrical skull we can find twelve or more points which lie in the mirror plane, while the "horizontal plane" has only some four points for its determination, and these in no case so simply determinable as the positions of those we have spoken of as mid-sagittal points. The "crux" lies in this: we have twelve or more points which "should" lie in one plane. If they don't, what is an adequate representative of that plane? The mathematician would answer at once that a "good" substitute for it would be the plane that made the sum of the squares of the distances from it of these mid-sagittal points a minimum‡. At the present stage of these investigations "weighting" the mid-sagittal points must be left on one side, and we solve our

* *Not* determined from the left Orbitale, but from the mean height of right and left Orbitalia.

† In very marked cases the unequal height of the ears is recorded as an "anatomical anomaly," see *Annals of Eugenics*, Vol. IV. pp. 235—6, Plates III and IV.

‡ Not infrequently such a plane is termed the "best" plane. But to accept this view we should first have to show that the deviations of these points from this plane followed the normal law of frequency, and that the standard errors of the deviations of the individual points were the same, otherwise the question of weighting the individual points arises. This suggests valuable, but very laborious work in determining the variations of what I have termed the mid-sagittal standard points from a mid-sagittal plane and a consideration of their individual frequency distributions. We should probably find ourselves finally thrust on the problem of whether the standard deviations varied from race to race. If they should do so, weighting would be very troublesome.

problem by stating that the standard mid-sagittal plane is the plane—in a mathematical sense—of close fit to the twelve or more mid-sagittal points.

It was with regard to problems of this kind, in particular cranial problems, that in 1901 I published a paper on "Lines and Planes of Closest Fit to Systems of Points in Space*," and showed that the determination of good fitting planes depended on finding the standard deviations and correlation coefficients of the coordinates of the points in space. The solution was reduced to a problem in solid Cartesian geometry. To apply it to the skull we must (i) determine some three rectangular planes associated with the skull—these I term the fundamental reference planes, and (ii) have some instrument which will rapidly provide us with the three coordinates of any point whatever of the skull in relation to these three planes. Such an instrument I term a cranial coordinatograph. One especially designed by me and made for me by Messrs Hawksley and Sons will be described below.

Now let us see where we stand. The skull may be looked upon as a system of indefinitely numerous points. By aid of the cranial coordinatograph we can at once form tables of the coordinates in space of any number of these points we please. The instrument enables us to construct plan and elevation drawings of these points. We can then proceed to deduce properties of the skull either by the methods of solid Cartesian geometry so familiar to the mathematician, or by the graphical rules of plan and elevation drawings so familiar to the engineer.

We are thus able to determine (i) the distance between any two points on the skull—the callipers may be dispensed with, although the tape will still be required; (ii) the equation to the line joining any two points and the angle made by this line with any other line or plane; (iii) the angle between any two planes as represented by their equations; (iv) the standard mid-sagittal plane as defined above, and—perhaps the most important of all determinations—whether (v) any two homologous points on the skull have true mirror symmetry. The cranial coordinatograph seems to me to throw open a new field in craniometry, much as the modern theory of statistics did some forty years ago. It adds solid analytical geometry to the technique of the craniologist, and provides a valuable addition to his *instrumentarium*.

I have no desire to screen the labour of calculation involved in the new processes suggested in this paper. The question is: Are the results worth that labour? Personally I think they are. It is something to gain a means whereby we can distinguish between the adequacy of various median sagittal planes, or obtain a measure of the inadequacy of the Frankfurt Horizontal Plane. And these are only two of the many problems solvable by the new method†.

* *Philosophical Magazine*, November, 1901, pp. 559—572.

† Apart from new problems we obtain other methods of solving old problems. We have the coordinates of Nasion, Alveolar Point and Basion; they give the equations to the sides of the fundamental facial triangle; and we can study its angular properties without using the trigonometer. Or again we have the coordinates of Nasion and Prosthion (or, if we prefer, Alveolar Point), and have the equation to their join. One of the direction cosines of this line gives the angle it makes with the Frankfurt Plane, that is the Profile Angle. We can thus dispense with the goniometer, which is at best a faulty instrument for it assumes that the line joining Nasion and Prosthion projects into a vertical line on the usual Transverse Vertical Plane, which in general it does not.

Before we can determine the coordinates of points on the skull we must fix on coordinate planes. These might of course be selected in any manner, but it is convenient to select for them already familiar planes. I call them the planes of reference, and the reader must distinguish them from the standard planes of the skull, which are something quite different. I take as planes of reference: (i) the Frankfurt Horizontal Plane as determined by balancing the Poria on the extreme points of the knife edges of a Ranke craniophor. The Orbitalia are then marked on right and left orbits and the skull rotated until the horizontal plane through the auricular axis (i.e. the join of the Poria* or the knife edge of the ear plugs) bisects the difference of the heights of the two Orbitalia†. This is our first reference plane. With the skull thus adjusted and the scriber at this height, the most posterior point in the occipital region in this plane (to be called the plane of $x = 0$) is marked on the skull. It will be called *Kappa*. This point Kappa is usually on or close to the occipital protuberance. The Frankfurt Horizontal Plane as thus defined will pass either through both Orbitalia or above one lower orbital margin and below the other. See Plate I (*a*). (ii) While the skull is still adjusted to this slight modification of the usual Frankfurt Plane, the horizontal bar of the craniophor is brought down until it is in contact with the skull and moved forward or backward until its point meets the sagittal suture. This point is marked on the skull. It is the Apex in my terminology‡. The plane through the Apex and the Poria is that of the so-called Transverse Vertical Plane, the plane for which the Transverse Contour is usually provided. It is to be our plane of (xz) or $y = 0$. The planes (i) and (ii) meet in the auricular axis which is accordingly our axis of z. (iii) *Any* plane perpendicular to the auricular axis will serve as the third plane of reference. Now suppose the skull removed from the craniophor, and adjusted so that the auricular axis is perpendicular to a drawing-board, then the paper on that drawing-board may be taken as the plane of (xy) or $z = 0$. The point in which the auricular axis meets this drawing-board will be our origin, or the origin is the plan of the two Poria. If we now project onto the drawing-board all points we please of the skull, we shall have their plans on the plane of $z = 0$. These will give us their x and y coordinates. The plan of the Apex joined to the plan of the Poria will give the axis of x, and the line in this plane through the plan of the Poria perpendicular to the join of the the plans of Apex and Poria will be the axis of y. This axis of y should *very* closely pass through the plan of Kappa if our adjustments have been accurately made. I take the positive direction of the axis of x to be away

* I prefer to mark the Poria only after the skull is supported on the knife edges.

† The scriber set to the level of the left of the knife edges is first applied to one Orbitale, and the skull rotated until this Orbitale and the Poria are in one plane. The scriber is then applied to the second orbit; if its Orbitale is *above* the scriber, the scriber point is marked on the skull below the Orbitale, and the skull rotated till the scriber bisects the difference. If the scriber as first set is above the second Orbitale, then that Orbitale is brought up to the scriber, which when applied to the first orbit will now be below its Orbitale and the bisection must take place on that orbit. The ultimate plane, to be called that of (zy) or $x = 0$, is our first plane of reference.

‡ The *Apex* must not be confused with the *Vertex*; the latter is the point of the skull at maximum perpendicular distance from the Frankfurt Horizontal Plane.

from the Apex, i.e. towards the base of the skull, and the positive direction of the axis of y away from Kappa or towards the face of the skull. I have chosen the plane of $y=0$, or the Transverse Vertical Plane, as the plane for giving the elevations. The Frankfurt Horizontal Plane is the plane through the axis of y perpendicular to the plane of the drawing board, and, if the skull were truly symmetrical, the plane of the plans would be parallel to the mirror plane or a true Median Sagittal Plane. Accordingly all the mid-sagittal points would have the same elevations.

Plates IV—VI provide photographs of the plan and elevation drawings of six skulls—those of a Fuegian, an ancient Egyptian from Nubia, a modern Arab, a Negro from the Teita Hills district, a 17th century Londoner, and a modern Hindu. These skulls were chosen at random, and the diagrams in every case show us that the mid-sagittal points do not lie in a single plane perpendicular to the auricular axis, for if they did these points would have equal elevations*. A true mid-sagittal plane is a fiction in every one of these cases, and we are compelled to replace it by the idea of the "closest fitting" plane to the chief mid-sagittal points.

2. *The Cranial Coordinatograph.*

In order to obtain the plans and elevations of points on the skull (not confining ourselves as in our present illustrations to the mid-sagittal points) it is needful to devise an instrument which will serve three purposes:

(*a*) Set the line joining any two selected points of the skull perpendicular to a drawing board. In our present illustration this line is the auricular axis or join of the Poria, but we might find in other investigations that some other line would be of more service as axis of z. This can equally well be achieved by the coordinatograph.

(*b*) The instrument must be capable of measuring the height of any point above the drawing-board, that is we must read off easily upon it the elevations of any chosen cranial points.

(*c*) At the same time it must by a simple action record on the drawing-board the plan of the cranial point. These objects are all achieved by the present instrument.

Diagrammatically the instrument consists of three arms, two of which are capable of fine motion and to each of which a vernier is attached. See Fig. 1 on p. 222.

AB is a vertical rod on which slide the two arms D_1P_1 and D_2P_2, which can be brought into absolute contact such that the points P_1 and P_2 coalesce. CD_3 is a fixed arm, and carries a vertical cylinder in the axis of which is a needle point P_3. When the button at C is pressed, this needle comes down on the drawing-board and the inked rim of the cylinder makes a circle with a needle point in its centre. This

* The models based upon the plan and elevation drawings provided by the cranial coordinatograph had to be tilted for photography in order that the names of the points should be legible. Accordingly the reader will find it best when examining them to hold the page nearly vertical, when the models come more closely into correct perspective.

is the plan of the point, with which one or other of the points P_1 or P_2 is placed in contact. The three points P_1, P_2 and P_3 are in a vertical line perpendicular to the base BB' of the instrument. Thus the line $P_1P_2P_3$ is accurately parallel to the vertical rod AB. This rod is graduated in millimetres from zero at B. Thus with the fine adjustments and verniers the height of P_1 or P_2 or their difference in height can be determined with close approximation. The other part of the instrument, the cranial staddle or skull trivet—it would be misleading or ambitious to call it a craniophor—consists of a triangular plate carrying a saucer, and supported by three screws at the angles. The skull is placed on a bed of plasticine in the saucer, with the required line roughly adjusted to the vertical. The point P_2 of the

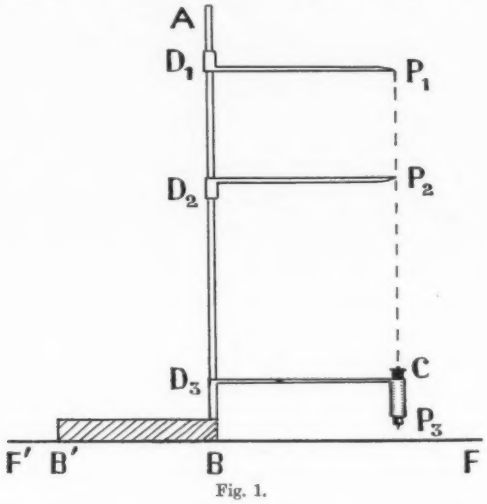


Fig. 1.

coordinatograph is brought into contact with the lower point of the line; the upper point P_1 is brought down to the approximate level of the upper point. Then by the turning of the three staddle screws and the fine adjustment movements of the arms D_1P_1 and D_2P_2, the skull is brought into contact at the given points—e.g. the Poria—with the terminals P_1 and P_2 of the arms of the coordinatograph. The adjustment does not take in practice much longer to carry out than it does to describe, and when it is completed the required line—e.g. the auricular axis—is perpendicular to the drawing-board—a touch on the button at C now records the position of the origin on the drawing-board. The upper arm is then raised above the skull out of the way, and the lower arm used; it is brought into contact successively with the points of the skull which are to have their coordinates found. The button C determines the plan, the scale on the vertical AB, with the vernier at D_2, gives the elevation of the point. The plan of the Apex gives the x-axis, and the line through the origin perpendicular to this gives the y-axis. Thus the three coordinates of any point can be found from the recorded elevations and the measurement of the plan drawing on the board.

The coordinatograph and the cranial staddle are made relatively heavy, so that they may not be moved too quickly either by over-haste or accident, and thus compel the user to readjust the skull and start afresh. Care has to be taken in adjusting the skull on its bed, that no required point lies immediately above or very close to one of the screw legs of the staddle. The base plate of the staddle is raised sufficiently above the drawing-board to allow of the plan-recorder D_3C passing beneath it as the plans of one or two points of the skull will frequently be beneath this plate. With practice in the use of the apparatus I think the adjustment of a skull and the determination of the three coordinates of some twenty points do not involve much more than an hour's labour.

Table I gives as an illustration the three coordinates of some fourteen points on six skulls referred to the reference planes we have discussed above. After a short discussion of mirror symmetry, I shall return to these sets of coordinates and indicate the type of problem in which their determination can be of service.

Plate VII (*a*) gives a photograph of the Cranial Coordinatograph in its first form. Slightly to the right we see a skull on the cranial staddle; to the left of it the two scriber arms of the coordinatograph are adjusted to the upper and lower Poria of the skull. The scale, the fine adjusting mechanisms of the scriber arms, with the verniers on the sloping faces of the cut away portions of the arm brackets are visible. Two instruments for recording plans are also in the picture. That on the right is of the Klaatsch type*, and is set for determining the plan of the glabella of the skull. That on the left is the plan-pricker from my osteometric instrument. The upper is the scriber arm, the lower arm marks the plan of its point by a circle (inked with a pad) with central needle point, corresponding to the cylinder-bearing arm of the diagram in Fig. 1. The use of such auxiliary instruments requires a double operation, the measurement of the elevation by the coordinatograph and the location of the plan by some form of vertical projector. This double instrumental setting has been got rid of by attaching an arm like that on my plan-pricker on the left to the base of the coordinatograph: see Plate VII (*b*). Thus, if either arm be set to a point on the adjusted skull, a tap on the plan-pricker gives the plan, and a reading of the scale on the vertical upright of the coordinatograph the elevation of the point in question.

3. *On the Symmetry of the Skull.*

I have indicated in my introductory remarks that I do not look upon the Frankfurt Horizontal Plane passing through the auricular axis as the best approach we can make to one of the fundamental planes of the skull. The first thing which strikes an observer of the human frame is its superficial approach to symmetry. This symmetry is not axial, but planar or mirror symmetry. Take any body whatever with a plane side and place that side against a mirror, and we see at once an example

* This does not differ essentially from Lissauer's Diagraph (*Archiv für Anthropologie*, Bd. xv. Supplement, S. 15 and Tafel XIV), but I was personally interested in Klaatsch's contour-tracings (of which I possess several originals), and I had the original projector used in the Biometric Laboratory made for me to his pattern by his instrument maker. I therefore speak of it as of "Klaatsch type" without claiming for him, or indeed for Lissauer, the invention of craniographs.

TABLE I.

Coordinates of Cranial Points (mms.).

Referred to Auricular Axis as axis of z and "Transverse" plane as plane of xz; $z=0$ is plane of drawing board.

Mid-sagittal Points*	(A) Fuegian			(B) Nubian			(C) Arab			(D) Negro			(E) English			(F) Hindu		
	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z
Alveolar Point ...	+ 60·8	+101·2	187·0	+ 44·6	+94·6	190·9	+ 46·6	+88·8	198·1	+ 41·5	+98·2	189·4	+ 42·8	+97·2	197·5	+ 37·8	+104·4	188·8
Nasal Spine	+ 36·9	+ 93·5	188·3	+ 24·1	+96·0	189·3	+ 28·8	+92·3	198·1	+ 23·5	+97·0	187·7	+ 27·4	+98·1	197·7	+ 25·3	+102·4	185·9
Nasion	− 20·5	+ 93·5	189·0	− 30·3	+92·8	189·3	− 29·4	+83·7	196·5	− 23·9	+93·0	187·8	− 30·5	+90·4	197·8	− 27·0	+ 89·6	184·2
Glabella	− 29·6	+ 96·9	188·6	− 41·0	+94·1	186·9	− 39·6	+84·8	196·0	− 35·7	+95·5	186·7	− 38·9	+94·5	196·8	− 36·3	+ 93·8	186·1
Bregma	−108·8	+ 20·4	185·8	−116·9	+10·4	190·1	−119·9	+ 4·1	190·6	−107·4	+24·2	183·1	−119·6	+16·4	199·1	−106·7	+ 10·9	188·6
Apex	−114·5	0·0	188·5	−118·3	0·0	189·7	−119·0	0·0	196·1	−112·1	0·0	184·3	−121·2	0·0	199·7	−107·6	0·0	187·4
Lambda	− 54·6	− 84·1	189·3	− 47·7	−95·4	189·1	− 49·9	−78·1	189·7	− 56·6	−82·1	185·0	− 66·0	−88·6	191·4	− 56·6	− 77·2	184·2
Kappa	0·0	− 89·0	190·5	0·0	−93·1	188·1	0·0	−71·2	189·8	0·0	−84·6	185·0	0·0	−86·9	198·0	0·0	− 82·2	185·4
Occipital Protuberance	+ 9·8	− 92·3	192·8	+ 16·4	−78·2	187·5	− 6·2	−75·5	190·6	+ 13·0	−73·5	187·4	+ 3·4	−83·0	197·0	+ 5·3	− 78·2	185·4
Inion (Inferior) ...	—	—	—	—	—	—	+ 6·9	−65·8	189·8	+ 13·9	−66·3	187·6	+ 8·1	−73·6	196·2	+ 8·5	− 69·3	186·8
Opisthion	+ 31·1	− 38·4	189·7	+ 30·0	−34·7	187·3	+ 29·8	−38·4	192·2	+ 25·9	−36·4	185·8	+ 17·3	−39·2	195·9	+ 26·9	− 41·5	186·6
Basion	+ 25·2	+ 1·6	187·4	+ 26·8	+ 2·9	187·9	+ 25·5	+ 4·1	193·1	+ 24·9	+ 1·6	185·4	+ 14·1	− 1·9	196·9	+ 24·7	− 0·5	188·7
Base of Palatal Spine	+ 38·0	+ 48·4	188·2	+ 28·4	+46·2	189·8	+ 30·8	+45·4	194·4	+ 28·8	+47·6	185·8	+ 21·3	+47·8	196·6	+ 27·7	+ 58·7	188·1
Mid-porion	0·0	0·0	190·0	0·0	0·0	189·85	0·0	0·0	194·8	0·0	0·0	186·2	0·0	0·0	196·0	0·0	0·0	186·0

* The Inferior Inion was not used in the case of (A) and (B). In the case of (A)—(F) the Kappa was so close to the x-axis that I have taken it on that axis.

of such mirror symmetry*. This apparent symmetry is the striking feature of the living head or the skull, and it is idle to neglect it and suppose we can determine any standard plane without regard to it. Of course as soon as we begin to measure we find that the skull is very far from symmetrical. Much work has been devoted in the Biometric Laboratory in recent years to the question of asymmetry in the human frame, and more will shortly be published. I do not propose to deal at present with the results of those investigations, although they actually upset some current beliefs. As far as the skull is concerned those investigations deal with the measurement of homologous distances or the size and shape of homologous bones. But we have to remember that the brain controls the development of the brain case as much or rather more than the case controls the brain. Let us start with a hypothetical brain of perfect mirror symmetry, and let it retain this symmetry from the earliest fetal life. The homologous bones will spread from their ossification centres over the brain but they will spread unequally and not homologously; the resulting brain case might possibly have mirror symmetry, but this could not be ascertained in general from a measurement of homologous bones. We need to ascertain first at least some approach to a probable mirror plane. If we suppose homologous bones to grow "at random," according to a general law, but that there be no absolute equality of growth in homologous bones in definite directions, then they will meet and the sutures be formed in a more or less random manner, first the edge of a right side, then of a left side bone protruding into the territory of the other. The best we can do is to take some form of average of the sutures which should lie in the mirror plane. As we cannot attempt this for every point on the whole series of sutures, we do the best we can by fitting a close plane to a reasonably large number of definite points on these sutures. If this gives us the best plane available for the skull, it by no means follows that it would be with equal closeness the mirror plane of the brain or of the living head. Indeed the brain might be truly symmetrical, while the skull was asymmetrical. Without having definite evidence to produce, I think, however, that the living head is on the average more asymmetrical than the skull. Suppose we take a full-face portrait of a head, how shall we determine what is its mirror plane, and judge what it would look like if symmetrical?

I tested in the first place a rough outline sketch of Cromwell's death mask—Plate VIII (ii) (*b*). The nose decidedly deviated to the left cheek. What is the "best" mirror plane? The only thing to be done was to bisect the lip line and the external occular distance; these points of bisection were joined, and the drawing cut in half down the bisecting line. The two halves were traced in reverse and then the two right sides were joined up and the two left sides were joined up to form absolutely symmetrical faces. The results in Plate VIII (ii) (*a*) and (*c*) are absurd, but suggestive. The reader can choose (ii) (*a*) with the duplication of the famous wart over the right eye, or (ii) (*c*) without the wart at all. We see at once that the skewness of the nose leads to a marked diminution of that

* The mirror must be a silvered plate, and not the ordinary glass mirror, for the latter will show a vacant sheet between object and image.

organ, or to its exaggeration. In Plate VIII (i) I have applied the same treatment to a drawing of the Ashmolean bust of Cromwell.

Now this difficulty follows us when we ask what would a familiar face look like were it symmetrical. We take a full-face portrait and at once are met by the problem: Where is the mirror plane to be placed? I took the photograph of a colleague and after several trials finally settled that again it was best to take the dichotomic line through the bisections of mouth and external orbital distance. This line would not be a truly vertical line as the head was slightly inclined to the right: see Plate IX (*b*). To my horror the bulk of the nose and neck fell in one half of the divided photograph. These two halves had now to be reversed, and this was done by rephotographing them, and obtaining two prints, one with the film against the printing paper and the other with the glass. Joining my two halves together I obtained two perfectly symmetrical faces—Plate IX (*a*) and (*c*)—but at the cost of much reality. Any one who examines these "symmetricised" portraits will I think come to the conclusions that (i) grace does not necessarily connote facial symmetry, and (ii) a good deal of personal individuality is linked with facial asymmetry.

After this I felt able to deal with the problem of a symmetrical skull. Precisely the same process was repeated on the *Norma facialis* of an Egyptian skull. See Plate X. Allowing for the weakening of the reversed prints, I think, any one accustomed to handling skulls will be struck by the naturalness and individuality of the central photograph as compared with the symmetricised products to right and left of it.

But while it is relatively easy to create symmetrical portraits of the living face or of the *Norma facialis* of a skull, it is far less easy to determine the plane associated with an actual head or skull, by aid of which we can appreciate its asymmetry. As I have indicated in the introduction to this paper, there are a certain number of points which may be termed the mid-sagittal points, for they should lie, if the skull were symmetrical, in one plane, the mid-sagittal (and are not infrequently assumed to do so even in a natural skull). These points are the following: (i) Alveolar Point, (ii) Nasal Spine, (iii) Nasion, (iv) Glabella, (v) Bregma, (vi) Apex, (vii) Lambda, (viii) Kappa (see p. 220 above), (ix) Occipital Protuberance (Superior Inion), (x) Inferior Inion, (xi) Opisthion, (xii) Basion, and (xiii) Palatal Spine. As I have indicated, the plane of closest fit, as judged by minimum mean square deviation to these points, will be defined as the standard mid-sagittal plane. In the past the mid-sagittal plane has been arbitrarily determined by selecting three of the mid-sagittal points, regardless of whether such plane was or was not at right-angles to other so-called standard planes. Let me illustrate the difficulties heretofore current by citing the weightiest authority on the subject. I am inclined to think that in his case as in the case of many others the man who writes the biggest book is held to be the weightiest authority. If we cannot form an impression on the reader by the lucidity of our writings, we can at least impress him by the weightiness of our volumes. The very origin of the term weighty for an authority may be studied from this aspect.

Rudolf Martin in his *Lehrbuch der Anthropologie in systematischer Darstellung* (note that word *systematic*!) tells us in Bd. II, S. 582, after emphasising the necessity of planes of orientation for the skull, that:

"Ueber die Mediansagittal Ebene kann kein Zweifel bestehen; sie ist durch drei Punkte (Nasion, Inion und Basion) bestimmt. Zwar liegen diese nicht immer genau in einer Ebene, aber die Abweichungen sind so unbedeutend, dass sie in der Praxis vernachlässigt werden können."

In translation:

There can be no doubt about the Median Sagittal Plane. It is determined by three points (Nasion, Inion and Basion). These points do not lie always exactly in one plane, but their deviations are so unimportant, that in practice they may be neglected.

Why Martin selected out of the many points which should lie on the mirror plane the Nasion, Inion and Basion "*without doubt*" as those to determine the Median Sagittal Plane, he does not tell us, and I cannot tell the reader. I do know that one of his three points is one of the most difficult to determine on the skull, and no two writers seem to agree on how it is to be found! But let us go a step further, we are told that the Median Sagittal Plane is to be determined by these three points—when they are found—that seems clear enough. But alas! we are then informed that the plane which passes through these three points will not exactly pass through them, but the deviations may be neglected. Did Martin not know that a plane is fully determined by three points? On S. 582, this would appear to be so; but on S. 583, speaking of the horizontal plane of the skull, he says it ought to go through four points, but as the skull is asymmetrical the plane can only be taken through *three*, "for three points mathematically determine a plane."

In the Biometric Laboratory the Nasion, Bregma and Lambda have been taken for drawing the median sagittal contour, partly because these points are more or less clearly determined by the intersection of the cranial sutures, and partly because the vault of the skull seems for many purposes more important than the base.

Now let us return for a moment to Martin. His median sagittal section is to be taken (notwithstanding negligible deviations) through the three points—Nasion, Inion and Basion. We naturally look up his definitions of these points. They are:

S. 619. The Nasion is the meeting point of the sutura nasofrontalis (i.e. the suture between the nasal and frontal bones) with the *Median Sagittal Plane*. This plane therefore determines *the Nasion*.

S. 615. The Inion is the point in which the Lineae nuciae superiores meet in the *Median Sagittal Plane*; if these lines are so feebly developed, that they do not reach the *Median Sagittal Plane*, they must be artificially produced till they do. Thus again the median sagittal plane determines the Inion.

S. 615. The Basion is the point in which the anterior border of the *foramen magnum* is met by the *Median Sagittal Plane*. Thus the most weighty of modern anthropologists defines the Median Sagittal Plane in terms of three cranial points, which according to him are only to be ascertained by an *a priori* knowledge of the Median Sagittal Plane. If this be "systematische Darstellung," is there not some need for a little mathematical logic—a little biometry to clear away these craniological fogs?

I have no desire to defend any particular plane which goes through three points as being the better representative of a plane which should pass through a dozen or more, but when we are told that there can be "kein Zweifel" as to what is the Median Sagittal Plane one is tempted to ask whether Martin's "ohne Zweifel" Plane is really superior to the nasion-bregma-lambda plane of the Biometric Laboratory. Having the coordinates of the thirteen mid-sagittal points for the six skulls selected at random which illustrate this paper, it was easy to write down the equations in those cases to Martin's Plane and the Biometric Laboratory Plane, and to measure (i) the angles between these planes, and the plane of "closest" fit to the thirteen points, and (ii) the mean square residuals of the mid-sagittal points from Martin's "ohne Zweifel" Plane and the Biometric Plane. The results are given in Table II, A and B.

TABLE II.

A. *Angles Plane of Maximum Symmetry makes with the Biometric and Martin's "Median Sagittal Planes."*

Skull	Biometric Laboratory Plane (Nasion, Bregma, Lambda)	Martin's "ohne Zweifel" Plane (Nasion, Inion, Basion)
Ancient Egyptian ...	0° 26′·6	1° 20′·0
Modern Arab ...	1° 36′·3	0° 55′·0
Negro (Teita) ...	1° 39′·5	5° 46′·7
Fuegian	1° 59′·6	7° 10′·5
17th century English	3° 14′·0	0° 47′·8
Bengal Hindu ...	3° 56′·7	6° 19′·2
Mean Angle	2° 8′·8	3° 43′·2

Or, Martin's Plane has on the average a 73 °/₀ increase of angular deviation on the Biometric Plane.

B. *Mean Square Residuals for the two "Median Sagittal Planes."*

Skull	Biometric Laboratory Plane	Martin's Plane
Ancient Egyptian ...	1·5167	1·5909
Modern Arab ...	4·0284	2·5998
Negro (Teita) ...	5·7284	32·0419
Fuegian	8·5585	62·4137
English	27·3563	4·0257
Bengal Hindu ...	31·1950	35·5764
Mean Value	13·0055	23·0414

Or, Martin's Plane has on the average a 77 °/₀ increase of Mean Square Residual on the Biometric Plane.

The sections A and B of Table II show us that neither the Biometric Laboratory Plane, nor Martin's "ohne Zweifel" Plane lies on the average very close to the

Plane of Minimum Deviation from the mid-sagittal points. This plane may be spoken of as the plane of nearest approach to the mirror plane of the skull or more shortly as the *Plane of Maximum Symmetry.* This latter plane we shall take as the First Standard Plane of the skull, or the Standard Median Vertical Plane. Naturally the Standard Horizontal Plane will be perpendicular to this plane, and we shall determine how far the Frankfurt Horizontal Plane is deficient in this respect. The Standard Transverse Vertical Plane will be again perpendicular to both our Standard Plane of Maximum Symmetry and to our Standard Horizontal Plane, and we shall determine how far this plane deviates from the usual Transverse Vertical Plane through the auricular axis perpendicular to the Frankfurt Horizontal. It will be seen that whether we judge by angular deviations from the planes of maximum symmetry or by the mean square deviations the nasion-bregma-lambda plane is on our present evidence much superior to the nasion-inion-basion plane. Accordingly we shall not consider it needful again to refer at length to the Median Sagittal Plane of Martin. When we speak of the "usual" Median Sagittal Plane, we shall mean that in which the long series of median sagittal contours issued by the Biometric Laboratory has been drawn, i.e. the nasion-bregma-lambda plane.

4. *Procedure for the Determination of the First Standard Plane or Plane of Maximum Symmetry of the Skull.*

I have already indicated the first stage of this procedure, the determination of the reference planes, and in particular the auricular axis which is to be set perpendicular to the plane of the drawing-board. But there is one point I should wish once more to emphasise. The auricular axis may be defined to be the line joining the extreme points of the knife edges on which the skull rests on the craniophor, *when these knife edges are properly adjusted.* This adjustment is sometimes defined as the process of bringing the tip of the knife edges to the *Poria.* But what are the Poria? Martin* defines the Porion "as that point on the upper border of the auricular passage which is vertically above the middle of the same." But how the middle of the auricular passage is to be found, he does not tell us, nor can I conceive how it is possible for an asymmetrical conichoidal space to have a "middle." Still less, if I could discover this "middle" could I take a "vertical" through it to meet the upper border of the auricular passage, because a vertical can only mean a line perpendicular to the horizontal plane, and that plane can only be found when the knife edge tips are already placed on the Poria. Thus according to Martin the Poria can only be found after the Frankfurt Horizontal Plane has already been determined. The very process of tilting the skull round on the knife edge tips to bring the Orbitalia to the height of the top of the knife edges causes the tips of the knife edges to slip along the upper border of the auricular passages. In my opinion the only way to determine the Poria is to mark them after the skull is adjusted on the craniophor to the Frankfurt Horizontal, the knife edges being withdrawn outwards to the very verges of the upper borders of the auricular passages. If, with some definition, other than Martin's, the Poria be marked before the Frankfurt Plane

* *Loc. cit.* S. 618.

is determined, then there is difficulty about successfully balancing the skull with its knife edge tips on these Poria; the tilting of the skull produces a constrained equilibrium and the knife edge tips tend to slip off the Poria and a minor catastrophe may result.

However, having marked in one way or another the Poria, it is fairly easy to adjust the skull by the three screws of the skull staddle so that the two Poria are in contact with two upper arm tips of the coordinatograph. A touch of the button marks the point, where the auricular axis meets the plane of the drawing-board, or is the plan of the Poria and the origin of coordinates. Before the coordinatograph is moved, the elevations of the Poria must be read off and recorded; a check on their accuracy is that their differences should equal the intraporial distance, which should be taken with the callipers, when the skull has been removed from the craniophor and before it is placed on the staddle. The Apex and the Kappa are now projected on to the drawing-board by aid of the coordinatograph, and their plans, joined to the plan of the Poria, give the axes of x and y respectively. These axes should be perpendicular. If they are not, the y-axis must be taken perpendicular to the x-axis, and the Kappa will have an x-coordinate differing from zero. Such a coordinate was too small to be measurable in the skulls dealt with by me. The plan and elevation of every other "mid-sagittal" point is obtained in the same manner, and the record for each skull will be a series of coordinates similar to those given in Table I.

The craniologist may not take it amiss if we remind him of the chief theorems in solid analytical geometry which are of value in the study of the skull. If (a_1, b_1, c_1) (a_2, b_2, c_2) (a_3, b_3, c_3) be the coordinates of three points, and

$$l_1x + m_1y + n_1z = p_1 \quad \text{and} \quad l_2x + m_2y + n_2z = p_2$$

be the equations to two planes in the prepared form, i.e. such that

$$l_1^2 + m_1^2 + n_1^2 = 1, \quad l_2^2 + m_2^2 + n_2^2 = 1,$$

then:

(i) The equation to the line passing through (a_1, b_1, c_1) (a_2, b_2, c_2) is

$$\frac{x - a_1}{a_2 - a_1} = \frac{y - b_1}{b_2 - b_1} = \frac{z - c_1}{c_2 - c_1} \quad \text{.....(i).}$$

(ii) The direction cosines of the lines are

$$\left.\begin{aligned} &L_{12} = (a_2 - a_1)/r_{12}, \quad M_{12} = (b_2 - b_1)/r_{12}, \quad N_{12} = (c_2 - c_1)/r_{12} \\ \text{where} \quad &r_{12}^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2 \end{aligned}\right\} \text{.....(ii).}$$

(iii) The angle between two lines, θ, is given by:

$$\cos\theta = L_{12}L_{34} + M_{12}M_{34} + N_{12}N_{34} \text{.....(iii).}$$

(iv) The angle ϕ between two planes as above is given by

$$\cos\phi = l_1l_2 + m_1m_2 + n_1n_2 \text{.....(iv).}$$

(v) The angle ψ between a line and a plane is given by

$$\sin\psi = L_{12}l_1 + M_{12}m_1 + N_{12}n_1 \text{.....(v).}$$

(vi) The plane through the points is given by

$$\begin{vmatrix} x-a_1 & y-b_1 & z-c_1 \\ a_2-a_1 & b_2-b_1 & c_2-c_1 \\ a_3-a_1 & b_3-b_1 & c_3-c_1 \end{vmatrix} = 0 \quad \text{.....(vi),}$$

where it is simplest to express the determinant numerically before expanding it.

(vii) Let x, y, z represent the coordinates referred to the three planes of reference already discussed of any of the "mid-sagittal" points and S denote a summation for all these points: let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the mean coordinates of all these points, n in number; let

$$\sigma_x^2 = \frac{1}{n} S(x^2) - \bar{x}^2, \quad \sigma_y^2 = \frac{1}{n} S(y^2) - \bar{y}^2, \quad \sigma_z^2 = \frac{1}{n} S(z^2) - \bar{z}^2,$$

be the x, y and z squared standard deviations; and let

$$p_{zy} = \frac{1}{n} S(zy) - \bar{z}\bar{y}, \quad p_{xz} = \frac{1}{n} S(xz) - \bar{x}\bar{z}, \quad p_{yx} = \frac{1}{n} S(yx) - \bar{y}\bar{x}$$

be the product moment coefficients.

We must then solve the equation

$$\begin{vmatrix} \sigma_x^2 - \Sigma^2, & p_{yx}, & p_{xz} \\ p_{yx}, & \sigma_y^2 - \Sigma^2, & p_{zy} \\ p_{xz}, & p_{zy} & \sigma_z^2 - \Sigma^2 \end{vmatrix} = 0 \quad \text{.....(vii),}$$

or

$$\Sigma^6 - \Sigma^4(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) + \Sigma^2(\sigma_z^2\sigma_y^2 + \sigma_x^2\sigma_z^2 + \sigma_y^2\sigma_z^2 - p^2_{zy} - p^2_{xz} - p^2_{yx}) - (\sigma_x^2\sigma_y^2\sigma_z^2 - \sigma_x^2 p^2_{zy} - \sigma_y^2 p^2_{xz} - \sigma_z^2 p^2_{yx} + 2p_{zy}p_{xz}p_{yx}) = 0 \quad \text{.....(viii),}$$

and we shall obtain three values Σ_1^2, Σ_2^2, Σ_3^2 of Σ^2*. These values of Σ^2 are the minimum and the two maximum values of the mean square deviations of the n points from three planes†. The Σ_1^2 which gives the minimum value is the one we are seeking in the first place; it provides the plane of maximum symmetry, or the nearest approach we can get to a true mirror plane or median sagittal plane of the skull. The Standard Horizontal Plane and the Standard Transverse Vertical Plane must be defined with regard to this plane, and we shall deal with them later on.

To determine this plane we must solve the equations:

$$\left.\begin{aligned} (\sigma_x^2 - \Sigma_1^2)L_1 + p_{yx}M_1 + p_{xz}N_1 &= 0 \\ p_{yx}L_1 + (\sigma_y^2 - \Sigma_1^2)M_1 + p_{zy}N_1 &= 0 \\ p_{xz}L_1 + p_{zy}M_1 + (\sigma_z^2 - \Sigma_1^2)N_1 &= 0 \end{aligned}\right\} \quad \text{.....(ix).}$$

subject to $L_1^2 + M_1^2 + N_1^2 = 1$

L_1, M_1, N_1 are the direction cosines of the plane of maximum symmetry, and

$$P_1 = L_1\bar{x} + M_1\bar{y} + N_1\bar{z} \quad \text{.....(x)}$$

or the equation of the plane is

$$L_1(x - \bar{x}) + M_1(y - \bar{y}) + N_1(z - \bar{z}) = 0 \quad \text{.....(xi)}$$

and Σ_1^2 measures the mean square deviation of the mid-sagittal points from this plane.

* *Philosophical Magazine*, 1901, pp. 561—563.

† All the terms in brackets are positive, and the equation has three real roots.

5. *Illustration of Numerical Work.*

We will now give a numerical example of finding a plane of maximum symmetry, and then more briefly provide the equations giving Σ_1^2 and this and other planes for several skulls. We will take the Arab skull, section (C) of Table I, p. 224, and consider the 13 sets of mid-sagittal points, coordinates given in the following table:

TABLE III.

Coordinates for Arab Skull.

Point	x	y	z	$z'=z-193{\cdot}5$
1. Alveolar Point ...	+ 46·6	+88·8	198·1	+4·6
2. Nasal Spine	+ 28·8	+92·3	198·1	+4·6
3. Nasion	− 29·4	+83·7	196·5	+3·0
4. Glabella	− 39·6	+84·8	196·0	+2·5
5. Bregma	−119·9	+ 4·1	190·6	−2·9
6. Apex	−119·0	0·0	196·1	+2·6
7. Lambda	− 49·9	−78·1	189·7	−3·8
8. Occipital Protuberance	− 6·2	−75·5	190·6	−2·9
9. Kappa	0·0	−71·2	189·8	−3·7
10. Inferior Inion ...	+ 6·9	−65·8	189·8	−3·7
11. Opisthion	+ 29·8	− 38·4	193·2	−0·3
12. Basion	+ 25·5	+ 4·1	193·1	− 0·4
13. Base Palatal Spine ...	+ 30·8	+45·4	194·4	+0·9
Sum	−195·6	+74·2	2516·0	+0·5

Sum ÷ 13:

(i) $\bar{x}=-15{\cdot}046{,}154$, $\bar{y}=+5{\cdot}707{,}692$, $\bar{z}=193{\cdot}538{,}4615$, $\bar{z}'=+{\cdot}038{,}4615$.

Squares:

(ii) $\bar{x}^2=226{\cdot}386{,}750$, $\bar{y}^2=32{\cdot}577{,}748$, ... $\bar{z}'^2={\cdot}001{,}479$.

Products:

(iii) $\bar{y}\bar{z}'=+{\cdot}219{,}526$, $\bar{z}'\bar{x}=-{\cdot}578{,}698$, ... $\bar{x}\bar{y}=-85{\cdot}878{,}813$.

Mean Squares of Coordinates from Table IV:

(iv) $S(x^2)/13=3002{\cdot}578{,}4615$, $S(y^2)/13=4259{\cdot}210{,}769$, $S(z'^2)/13=9{\cdot}540{,}769$.

Mean Products from Table IV:

(v) $S(yz')/13=181{\cdot}356{,}923$, $S(z'x)/13=29{\cdot}892{,}308$, $S(xy)/13=365{\cdot}822{,}308$.

(iv) minus (ii): $\sigma_x^2=2776{\cdot}191{,}711$, $\sigma_y^2=4226{\cdot}633{,}021$, $\sigma_z^2=9{\cdot}539{,}290$.

(v) minus (iii): $p_{yz}=181{\cdot}137{,}397$, $p_{zx}=30{\cdot}471{,}006$, $p_{xy}=451{\cdot}701{,}121$.

We are now in a position to write down the fundamental cubic as given by Equation (viii). It is

$$\Sigma^6-7012{\cdot}364{,}022\,\Sigma^4+1156{,}2972{\cdot}392{,}766\,\Sigma^2-1996{,}0121{\cdot}046{,}469=0.$$

The ratio of the coefficients of the last two terms is 1·726, and we test with $\Sigma^2=1{\cdot}730$ giving 22839, and 1·728 giving − 238·353. The root is close to 1·728.

Applying Newton's Rule we obtain 1·728,0203 and thence finally $\Sigma_1^2 = 1{\cdot}728{,}0207$. This is the mean square deviation of the thirteen mid-sagittal points from the Plane of Maximum Symmetry. The other two roots of the cubic will be discussed later.

TABLE IV.

Squares and Products of the Coordinates of the Arab Skull.*

	x^2	y^2	z'^2	yz'	$z'x$	xy
(1)	+2171·56	7885·44	21·16	+408·48	+214·36	+4138·08
(2)	829·44	8519·29	21·16	+424·58	+132·48	+2658·24
(3)	864·36	7005·69	9·00	+251·10	− 88·20	−2460·78
(4)	1568·16	7191·04	6·25	+212·00	− 99·00	−3358·08
(5)	14376·01	16·81	8·41	− 11·89	+347·71	− 491·59
(6)	14161·00	0·00	6·76	0·00	−309·40	0·00
(7)	2490·01	6099·61	14·44	+296·78	+189·62	+3897·19
(8)	38·44	5700·25	8·41	+218·95	+ 17·98	+ 468·10
(9)	0·00	5069·44	13·69	+263·44	0·00	0·00
(10)	47·61	4329·64	13·69	+243·46	− 25·53	− 454·02
(11)	888·04	1474·56	·09	+ 11·52	− 8·94	−1144·32
(12)	650·25	16·81	·16	− 1·64	− 10·20	+ 104·55
(13)	948·64	2061·16	·81	+ 40·86	+ 27·72	+1398·32
Sum	39033·52	55369·74	124·03	+2357·64	+388·60	+4755·69
Sum ÷ 13	3002·578,4615	4259·210,769	9·540,769	+181·356,923	+29·892,308	+365·822,308

The whole of these values (Tables III and IV) have been put down for the use of any craniologist who may desire to test the labour of finding a Plane of Maximum Symmetry.

We have now to determine the direction cosines L_1, M_1, N_1, of the Plane of Maximum Symmetry from Equations (ix). These give us:

$$2774{\cdot}463{,}690\,L_1 + 451{\cdot}701{,}121\,M_1 + 30{\cdot}471{,}006\,N_1 = 0,$$

$$451{\cdot}701{,}121\,L_1 + 4224{\cdot}905{,}000\,M_1 + 181{\cdot}137{,}397\,N_1 = 0,$$

$$30{\cdot}471{,}006\,L_1 + 181{\cdot}137{,}397\,M_1 + 7{\cdot}811{,}269\,N_1 = 0.$$

From the first two equations we find

$$\frac{L_1}{1} = \frac{M_1}{10{\cdot}418{,}268} = \frac{-N_1}{245{\cdot}492{,}616}$$

for the relative values, and since $L_1^2 + M_1^2 + N_1^2 = 1$ we have for the absolute values of the direction cosines

$$L_1 = {\cdot}0040{,}6975, \quad M_1 = {\cdot}0423{,}9970, \quad N_1 = -{\cdot}9990{,}9244,$$

and if these values be substituted in the third equation for L_1, M_1, N_1 above, the

* In actual practice the squares are put directly on the machine from Barlow's Tables, and the product multiplications are a continuous process on the machine.

left-hand side will be found to be + ·0000,0087 instead of zero, which checks the value for Σ_1^2 as we have only worked to six decimal places in the coefficients.

Finally substituting in the values for P_1 given by (x) we find $P_1 = -193{\cdot}181,851$ or the equation to the Plane of Maximum Symmetry is

$$-{\cdot}0040,6975x - {\cdot}0423,9970y + {\cdot}9990,9244z = +193{\cdot}181,851 \quad \ldots\ldots(xii).$$

Knowing this plane we can determine the angles it makes with:

(*a*) the Frankfurt Horizontal Plane, i.e. $x = 0$. The cosine of this angle $= -{\cdot}0040,6975$, or the angle is 90° 14′·0. Thus the Frankfurt Horizontal Plane is not for this skull perpendicular to the Plane of Maximum Symmetry;

(*b*) the Transverse Vertical Plane, i.e. $y = 0$. The cosine of this angle $= -{\cdot}0423,9970$ and the angle is 90° 21′·8, or the Transverse Vertical Plane is not perpendicular to the Plane of Maximum Symmetry;

(*c*) the Plane perpendicular to the auricular axis, i.e. $z = 0$. The cosine of this angle is ·9990,9244 and the angle is 2° 26′·5. Thus the auricular axis for this skull is not perpendicular to the best plane we can adopt for mid-sagittal symmetry.

Again for this skull the Left Porion has for coordinates (0, 0, 255·7) and the Right Porion (0, 0, 133·9); thus the Interporial Distance equals 121·8, and the Mid-porion is (0, 0, 194·8). Thus the Mid-porion is 1·3 mm. above the mean height of the thirteen mid-sagittal points, and further the Plane of Maximum Symmetry does not pass through the Mid-porion, but meets the auricular axis at the point $z = 193{\cdot}36$. The perpendicular distances of the Poria from the Plane of Maximum Symmetry are Left Porion 62·29 and Right Porion 59·40, which indicate the extent the two ears are asymmetrically placed. We can now write down the Equation for the Median Sagittal Plane as based on Nasion, Bregma and Lambda for this skull. By Equation (vi) it is

$$\begin{vmatrix} x + 119{\cdot}9, & y - 4{\cdot}1, & z - 190{\cdot}6 \\ -29{\cdot}4 + 119{\cdot}9, & 83{\cdot}7 - 4{\cdot}1, & 196{\cdot}5 - 190{\cdot}6 \\ -49{\cdot}9 + 119{\cdot}9, & -78{\cdot}1 - 4{\cdot}1, & 189{\cdot}7 - 190{\cdot}6 \end{vmatrix} = 0,$$

or,

$$\begin{vmatrix} x + 119{\cdot}9, & y - 4{\cdot}1, & z - 190{\cdot}6 \\ 90{\cdot}5, & 79{\cdot}6, & 5{\cdot}9 \\ 70{\cdot}0, & -82{\cdot}2, & -0{\cdot}9 \end{vmatrix} = 0,$$

or, expanding,

$$413{\cdot}34x + 494{\cdot}45y - 13011{\cdot}10z + 252,7447{\cdot}881 = 0.$$

Dividing by the square root of the sum of the squares of the coefficients of x, y, z we have the equation to the plane in its prepared form, i.e. with the coefficients the direction cosines and the constant term the perpendicular from the origin,

$$-{\cdot}031,7294x - {\cdot}037,9556y + {\cdot}998,7756z = +194{\cdot}015,354.$$

It is possible now by inserting the coordinates of any of the mid-sagittal points to determine its distance from this plane. The following are the distances of these points for the Arab skull:

(1) Alveolar Point	+1·0070	(8) Occipital Protuberance	+ ·5864
(2) Nasal Spine...	+ ·5750	(9) Kappa	+1·7453
(3) Nasion ...	·0000	(10) Inferior Inion ...	+2·1692
(4) Glabella ...	+ ·2175	(11) Opisthion	+ ·5399
(5) Bregma ...	·0000	(12) Basion...	+2·1165
(6) Apex... ...	−5·6204	(13) Base Palatal Spine ...	+2·5538
(7) Lambda ...	·0000	Sum of Squares of these Distances	52·3686

Mean Square Distance from Plane 4·0284.

Thus although the plane actually passes through *three* of the mid-sagittal points, its mean square deviation is more than double that of the Plane of Maximum Symmetry. While the mid-sagittal contour drawn in the Nasion-Bregma-Lambda Plane may serve many useful comparative purposes it clearly differs widely from any suitable mirror plane, and actually for this Arab skull the angle between this $\nu.\beta.\lambda.$ plane and the Plane of Maximum Symmetry is 1° 36′·3. It makes an angle of 91° 49′·1 instead of a right angle with the Frankfurt Horizontal Plane, an angle of 92° 50′·5 with the auricular axis and an angle of 92° 10′·5 with the Transverse Vertical Plane.

We may give as one furthur illustration the measure of prognathism as found from the angle between the Frankfurt Horizontal Plane and the line joining Alveolar Point to Nasion*. These points are (1) and (3), or for the Arab skull (+ 46·6, + 88·8, + 198·1) and (− 29·4, + 83·7, + 96·5). The equation to the line joining them is

$$\frac{x-46{\cdot}6}{-29{\cdot}4-46{\cdot}6}=\frac{y-88{\cdot}8}{83{\cdot}7-88{\cdot}8}=\frac{z-198{\cdot}1}{196{\cdot}5-198{\cdot}1},$$

or

$$\frac{x-46{\cdot}6}{76{\cdot}0}=\frac{y-88{\cdot}8}{5{\cdot}1}=\frac{z-198{\cdot}1}{1{\cdot}6},$$

or the direction cosines (l, m, n) of this line are proportional to

$$76{\cdot}0, \qquad 5{\cdot}1, \qquad 1{\cdot}6,$$

giving for absolute value, since $l^2+m^2+n^2=1$,

$$+\,{\cdot}997{,}536, \qquad +\,{\cdot}066{,}940, \qquad +\,{\cdot}021{,}001.$$

The Sine of the angle this lines makes with the Frankfurt Horizonal Plane, i.e. $x=0$, is ·997,536 or the Profile Angle itself = 85° 58′·6.

If we ask how far does the Nasion-alveolar line lie outside the Plane of Maximum Symmetry we have to find the angle between the plane whose direction cosines are

$$-\,{\cdot}004{,}070, \qquad -\,{\cdot}042{,}400, \qquad +\,{\cdot}999{,}092,$$

and the line with direction cosines

$$+\,{\cdot}997{,}536, \qquad +\,{\cdot}066{,}940, \qquad +\,{\cdot}021{,}001.$$

* This is of course the measure of prognathism in the skull, but the racial order of prognathism in the living, owing to the thickness and protrusion of the lips, may be very different. Perhaps this fact is not always adequately emphasised.

The sine of the angle (see Equation (v)) is + ·014,0837, giving an angle of + 48′·4, or the Profile Line is skewed out of the Plane of Maximum Symmetry towards the right side by more than three-quarters of a degree.

We have illustrated sufficiently the general algebraic and numerical processes by which, when the cranial coordinatograph has given the coordinates, the properties of any skull can be discussed by the aid of the elementary formulae of analytical geometry of three dimensions.

6. *Analytical Geometry of six illustrative Crania.*

The following six illustrative skulls were chosen at random. It is not suggested that average results obtained from them would not be widely modified when considerable series of the same races are dealt with; they are solely given here as indicative of the type of problems the cranial coordinatograph enables the craniologist to attack.

We have the following numerical results:

(*A*) *Fuegian Skull** (Recent).

Elevation of R. Porion 253·8, L. Porion 126·2, Mid-porion 190·0, Bi-porionic Distance 127·6.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6 - 7971{\cdot}463{,}075\,\Sigma^4 + 1460{,}0879{\cdot}769{,}528\,\Sigma^2 - 2446{,}0978{\cdot}059{,}113 = 0.$$

Value of $\Sigma_1^2 = 1{\cdot}676{,}844$.

Plane of Maximum Symmetry:

$$-\,{\cdot}0095{,}7040\,x + {\cdot}0157{,}3352\,y + {\cdot}9998{,}3036\,z = +\,189{\cdot}0220{,}8197.$$

$\nu.\beta.\lambda$. Plane†:

$$-\,{\cdot}044{,}7048\,x + {\cdot}010{,}2709\,y + {\cdot}998{,}9474\,z = +\,190{\cdot}677{,}8475.$$

Martin's Plane‡:

$$+\,{\cdot}122{,}4878\,x + {\cdot}038{,}6513\,y + {\cdot}992{,}9011\,z = +\,188{\cdot}966{,}1990.$$

(*B*) *Nubian Skull** (Ancient Egyptian from Kerma).

Elevation of R. Porion 247·0, L. Porion 132·7, Mid-porion 189·85, Bi-porionic Distance 114·3.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6 - 7662{\cdot}666{,}546\,\Sigma^4 + 1392{,}3838{\cdot}355{,}612\,\Sigma^2 - 1801{,}6485{\cdot}994{,}210 = 0.$$

Value of $\Sigma_1^2 = 1{\cdot}294{,}8535$.

Plane of Maximum Symmetry:

$$+\,{\cdot}005{,}4027\,x - {\cdot}006{,}2292\,y + {\cdot}999{,}9660\,z = +\,188{\cdot}664{,}6436.$$

* In the case of these crania the Inferior Inion was omitted.

† The usual "mid-sagittal" plane, i.e. that through Nasion (ν), Bregma (β) and Lambda (λ).

‡ The plane through Nasion, Inion, Basion, assumed by Martin to be "ohne Zweifel" the "best" mid-sagittal plane.

$\nu.\beta.\lambda.$ Plane:

$$-\cdot0112{,}3688\,x+\cdot0021{,}0153\,y-\cdot999{,}9347\,z=+188\cdot752{,}1300.$$

Martin's Plane:

$$+\cdot013{,}9371\,x-\cdot006{,}7188\,y+\cdot999{,}8803\,z=188\cdot231{,}5380.$$

(*C*) *Arab Skull* (Modern from Palestine).

Elevation of R. Porion 133·9, L. Porion 255·7, Mid-porion 194·8, Bi-porionic Distance 121·8.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7012\cdot364{,}022\,\Sigma^4+1156{,}2972\cdot392{,}766\,\Sigma^2-1996{,}0121\cdot046{,}469=0.$$

Value of $\Sigma_1^2=1\cdot7280{,}2066$.

Plane of Maximum Symmetry:

$$-\cdot0040{,}6975\,x-\cdot0423\cdot9970\,y+\cdot9990{,}9244\,z=+193\cdot181{,}851.$$

$\nu.\beta.\lambda.$ Plane:

$$-\cdot031{,}7294\,x-\cdot037{,}9556\,y+\cdot998{,}7756\,z=+194\cdot015{,}354.$$

Martin's Plane:

$$-\cdot010{,}3855\,x+\cdot035{,}5214\,y-\cdot999{,}3150\,z=-193\cdot086{,}908.$$

(*D*) *Teita Negro Skull.*

Elevation of R. Porion 130·6, L. Porion 241·8, Mid-porion 186·2, Bi-porionic Distance 111·2.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7350\cdot914{,}793\,\Sigma^4+1196{,}5899\cdot9308\,\Sigma^2-1369{,}2532\cdot1372=0.$$

Value of $\Sigma_1^2=1\cdot1451{,}0147$.

Plane of Maximum Symmetry:

$$-\cdot0220{,}1936\,x-\cdot0083{,}0922\,y+\cdot9997{,}2301\,z=+186\cdot384{,}314.$$

$\nu.\beta.\lambda.$ Plane:

$$-\cdot0508{,}8478\,x-\cdot0064{,}6706\,y+\cdot9986{,}8359\,z=+188\cdot167{,}4885.$$

Martin's Plane:

$$+\cdot076{,}1587\,x+\cdot014{,}4832\,y+\cdot996{,}9905\,z=+186\cdot761{,}565.$$

(*E*) *17th Century English Skull* (St Bride's Graveyard).

Elevation of R. Porion 258·7, L. Porion 133·3, Mid-porion 196·0, Bi-porionic Distance 125·4.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7887\cdot700{,}659\,\Sigma^4+1383{,}3943\cdot158{,}623\,\Sigma^2-4033{,}3720\cdot552{,}071=0.$$

Value of $\Sigma_1^2=2\cdot9204{,}2324$.

Plane of Maximum Symmetry:

$$+\cdot0081{,}2136\,x-\cdot0109{,}6533\,y+\cdot9999{,}0690\,z=+196\cdot739{,}7800.$$

$\nu.\beta.\lambda.$ Plane:

$$+\cdot0528{,}8718\,x-\cdot0461{,}5479\,y+\cdot9975{,}3330\,z=+191\cdot526{,}634.$$

Martin's Plane:

$$+\cdot0178{,}5293\,x-\cdot0011{,}2259\,y+\cdot9998{,}3999\,z=+197\cdot122{,}354.$$

(*F*) *Hindu Skull* (Modern, Bengal).

Elevation of R. Porion 130·4, L. Porion 241·6, Mid-porion 186·0, Bi-porionic Distance 111·2.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7399\cdot968{,}994\,\Sigma^4+1162{,}8428\cdot712{,}714\,\Sigma^2-2607{,}0720\cdot672{,}768=0.$$

Value of $\Sigma_1^2=2\cdot2451{,}8824$.

Plane of Maximum Symmetry:

$$-\cdot0017{,}4154\,x-\cdot0054{,}1230\,y+\cdot9999{,}8384\,z=+186\cdot605{,}594.$$

$\nu.\beta.\lambda.$ Plane:

$$+\cdot0667{,}4424\,x-\cdot0118{,}7277\,y+\cdot9976{,}9948\,z=+180\cdot915{,}098.$$

Martin's Plane:

$$-\cdot1114{,}3673\,x-\cdot0143{,}7877\,y+\cdot9936{,}6750\,z=+184\cdot759{,}760.$$

On the basis of these and similar results we will now proceed to some comparisons.

7. *Angular Relations.*

We have seen that it has been customary to treat the Frankfurt Horizontal Plane, the Transverse Vertical Plane through the auricular axis and a certain plane termed the Median Sagittal Plane as standard planes of the skull. Such standard planes should be mutually rectangular, but while the first two are at right angles, they are rarely perpendicular to the third. The third plane as defined by Martin, to judge from the present illustrative crania, seems very inferior to the $\nu.\beta.\lambda.$ plane (see our p. 228). We shall here then confine our attention to the latter plane. Table V provides the angles between our First Standard Plane of the skull—the Plane of Maximum Symmetry—as representing the mirror plane of the skull, and the planes which have been usually hitherto treated as standard planes, but which we treat merely as planes of reference.

Section (α) of the Table shows that the $\nu.\beta.\lambda.$ Mid-sagittal Plane in none of the six skulls approaches closely to the Plane of Maximum Symmetry, the average angle between the two planes being more than two degrees. In the English skull there is more than three degrees and in the Hindu nearly four degrees. Thus the $\nu.\beta.\lambda.$ plane cannot be looked upon as a close fit to the mid-sagittal points.

Section (β) shows us that the Frankfurt Horizontal Plane is *not* perpendicular to the Plane of Maximum Symmetry, which I personally think should be a prerequisite of a standard horizontal plane. The average deviation is over 30′.

TABLE V.

TABLE V.

Angles between Standard Planes and Planes of Reference.

Angle between:	Ancient Egyptian (Nubia)	Arab (Modern)	Negro (Teita)	Fuegian	English (17th century)	Hindu (Bengal)
(α) Plane of Maximum Symmetry and "Median Sagittal" Plane ($\nu.\beta.\lambda.$)	0° 24′·6	1° 36′·3	1° 39′·5	1° 59′·6	3° 14′·0	3° 56′·7

These planes may differ widely, almost visibly from one another. Average Angle 2° 8′·45.

	Negro (Teita)	Fuegian	Arab (Modern)	Hindu (Bengal)	Ancient Egyptian (Nubia)	English (17th century)
(β) Plane of Maximum Symmetry and Frankfurt Horizontal Plane	91° 15′·7	90° 35′·7	90° 14′·0	90° 6′·0	89° 41′·4	89° 31′·9

Average Deviation from perpendicularity = 30′·3

	Hindu (Bengal)	Negro (Teita)	Fuegian	Arab (Modern)	Ancient Egyptian (Nubia)	English (17th century)
(γ) "Mid-sagittal" Plane ($\nu.\beta.\lambda.$) and Frankfurt Horizontal Plane	93° 49′·6	92° 55′·0	92° 34′·3	91° 49′·1	89° 21′·4	86° 58′·1

Average Deviation from perpendicularity = 2° 27′·3

	Hindu (Bengal)	Ancient Egyptian (Nubia)	English (17th century)	Fuegian	Negro (Teita)	Arab (Modern)
(δ) Plane of Maximum Symmetry and Plane perpendicular to Auricular Axis	0° 19′·6	0° 28′·8	0° 47′·0	1° 5′·3	1° 20′·0	2° 26′·5

Average Angle between Planes = 1° 4′·5

	Ancient Egyptian (Nubia)	Fuegian	Arab (Modern)	Negro (Teita)	Hindu (Bengal)	English (17th century)
(ϵ) "Mid-sagittal" Plane ($\nu.\beta.\lambda.$) and Plane perpendicular to Auricular Axis	0° 39′·3	2° 37′·7	2° 50′·5	2° 56′·4	3° 53′·2	4° 1′·5

Average Angle between Planes = 2° 49′·8

	English (17th century)	Negro (Teita)	Arab (Modern)	Ancient Egyptian (Nubia)	Hindu (Bengal)	Fuegian
(ζ) Plane of Maximum Symmetry and Transverse ("Vertical") Plane	90° 37′·7	90° 28′·5	90° 21′·8	90° 21′·2	90° 18′·6	89° 5′·4

Average Deviation from perpendicularity = 30′·4

	English (17th century)	Arab (Modern)	Hindu (Bengal)	Ancient Egyptian (Nubia)	Negro (Teita)	Fuegian
(η) "Mid-sagittal" Plane ($\nu.\beta.\lambda.$) and Transverse ("Vertical") Plane	92° 38′·7	92° 10′·5	90° 40′·8	90° 7′·2	89° 37′·8	89° 24′·7

Average Deviation from perpendicularity = 1° 5′·8

Section (γ) indicates that the usual ($\nu.\beta.\lambda.$) Mid-sagittal Plane is even worse in the degree of perpendicularity to the Frankfurt Horizontal Plane, the average deviation amounting to $2° 27'·3$ or more than four times that of the Plane of Maximum Symmetry.

Section (δ) shows us that the auricular axis (Bi-porionic line) is not perpendicular to the Plane of Maximum Symmetry, the average deviation for the six crania being slightly over 1°. From the standpoint of the present writer this is one measure of the asymmetrical location of the ears. Relative to a good approximation to a mirror plane the two ears are shifted slightly forward or backward, upward or downward.

Section (ϵ) gives the like angle between the auricular axis and the perpendicular to the $\nu.\beta.\lambda.$ Plane. The average angle for this skull is $2° 49'·8$ or more than $2\frac{1}{2}$ times as great as in the case of the Plane of Maximum Symmetry.

Section (ζ) gives the angle between the Transverse Vertical Plane and that of Maximum Symmetry, the average deviation is half a degree, or the customary transverse vertical plane is not perpendicular to the plane of closest fit to the mid-sagittal points, as it should be in the opinion of the present writer.

Section (η) indicates, however, that the Plane of Maximum Symmetry is more than twice as fit as the $\nu.\beta.\lambda.$ Plane to represent the mid-sagittal section. If we start from the Frankfurt Plane and the Transverse Vertical Plane passing through the auricular axis as standard rectangular planes of the skull, then the Plane of Maximum Symmetry is more nearly perpendicular to both of these than the usual mid-sagittal section.

8. *Mean Square Deviation of the Mid-sagittal Points from various Planes.*

We can now look at another aspect of the relationship of the mid-sagittal points to the various planes which may be suggested for the mid-sagittal section. We may determine the Mean Square Distance, or the so-called Mean Square Residual, of the mid-sagittal points from the various planes. These values are given in Table VI. The five planes which may be considered as possible mid-sagittal planes are: (*a*) Our standard plane, the Plane of Maximum Symmetry: this must, of course, have the minimum Mean Square Distance. (*b*) The plane perpendicular to the auricular axis at the mean elevation, $\bar{z}$, of the mid-sagittal points. We may term this the "$\bar{z}$ Plane"; its Mean Square Distance is the σ_z^2 of our p. 231. (*c*) The plane perpendicular to the auricular axis and bisecting it. This may be termed the mid-porion sagittal plane. (*d*) The mid-sagittal plane through Nasion, Bregma and Lambda (the $\nu.\beta.\lambda.$ Plane). (*e*) Martin's Plane through Nasion, Inion and Basion.

We have already (see p. 228) discussed the last two planes, (*d*) and (*e*) of the Table. As far as the present skulls are concerned neither is comparable with (*a*), (*b*) and (*c*). If any craniologist finds the solving of a cubic equation too severe a mathematical labour, the coordinatograph will rapidly give him $\bar{z}$ and the Mean Elevation Mid-sagittal or z-Plane. The Plane of Maximum Symmetry is twice as good

as this plane, but the latter is very much better than either of the planes fixed by three mid-sagittal points only.

Clearly the mean square residual for any skull is a rough test of how far the mid-sagittal points lie in one plane, i.e. how far there is a true mid-sagittal plane, on the basis of which we could test the mirror symmetry of other points of the skull. A little thought, however, shows that this is only a rough approximation; the skulls are of different absolute sizes, and if we increased the linear dimensions of a skull by 5 °/₀ we should increase the mean square residual by more than 10 °/₀. We need accordingly an index from which we have eliminated the absolute size of the skull. We cannot therefore assert that the relative mid-sagittal asymmetry of the above six skulls is measured by the numbers in the (*a*) column of Table VI.

TABLE VI.

Mean Square Distance of the Mid-sagittal Points from five Planes which may be treated as Mid-sagittal.

Skull	(*a*) Plane of Maximum Symmetry as Standard Mid-sagittal Plane	(*b*) Plane perpendicular to Auricular Axis at Mean Elevation of Mid-sagittal Points	(*c*) Plane perpendicular to Auricular Axis through Mid-porion	(*d*) Usual Mid-sagittal Plane, or Plane through Nasion, Bregma and Lambda	(*e*) Martin's Mid-sagittal Plane, or Plane through Nasion, Inion and Basion
Negro (Teita) ...	1·1451	2·7498	2·7593	5·7284	32·0419
Ancient Egyptian (Nubia)	1·2949	1·5945	2·4443	1·5167	1·5909
Fuegian	1·6768	2·8727	4·8327	8·5585	62·4137
Arab (Modern) ...	1·7280	9·5393	11·1308	4·0284	2·5998
Hindu (Bengal) ...	2·2452	2·4083	2·8062	31·1950	35·5764
English (17th century) ...	2·9204	3·6529	4·5923	27·3563	4·0257
Mean for six skulls	1·8351	3·7912⁵	4·7609	13·0055	23·0414

To allow for the absolute size of the skull I have taken three lengths of the skull in directions at right angles, choosing these lengths from the plan and elevation drawings rather than from calliper measurements of the skull. I have taken:

(*a*) The Bi-porionic Distance; this is a length on the axis of z.

(*b*) The perpendicular from the Apex on the auricular axis. This is the x coordinate of the Apex.

(*c*) The length of the projection of the line joining Nasion to Kappa onto the axis of y, i.e. onto the line joining the plan of Kappa to the plan of the Poria.

I have then squared the cube root of the product of (*a*), (*b*) and (*c*), and thus obtained a quantity depending on the squared linear dimensions of the skull, by which it seemed that the mean square residuals might be divided so as to obtain a reasonable index independent of the dimensions of the skull. The resulting

number has been multiplied by 100. In the case of our six skulls the following Table gives the resulting indices.

TABLE VII.

Indices of Mesial Asymmetry.

Skull	(α) Minimum Mean Square Residual	(β) Product of the Cranial Rectangular Lengths $(a)\times(b)\times(c)$	(γ) $\frac{1}{4}(a)\times(b)\times(c)^{\frac{2}{3}}$	(δ) $\frac{10^6(\alpha)}{(\gamma)}$ = Index
Negro (Teita)	1·1451	2,075,509·080	16271·1	70·38
Ancient Egyptian (Nubia)	1·2949	2,513,682·171	18487·3	70·04
Fuegian	1·6768	2,664,900·480	19220·5	87·24
Arab (Modern)	1·7280	2,245,151·580	17146·0	100·78
Hindu (Bengal)	2·2452	2,050,821·568	16141·8	139·09
English (17th century) ...	2·9204	2,694,690·504	19364·5	149·88

Only one change has been made in the order of the crania by reducing the minimum mean square residuals to an index independent of absolute size. It would be foolish to draw any conclusion as to the relative position of the races to which these six skulls belong from the order of either column (α) or (δ) of this Table, but the Table may suggest interesting problems, which might be followed up, if considerable racial groups were worked out*. I have contented myself here by showing the manner in which numerical results may be obtained.

9. *The remaining Standard Planes.*

If we accept the view that the Plane of Maximum Symmetry is the most suitable plane to take as a mid-sagittal plane, and we term it our First Standard Plane, the question immediately follows: How are we to select our remaining standard planes? These will correspond respectively to the Horizontal Plane and Transverse Vertical Plane of the skull, but they must be chosen so as to be at right angles to one another and to the Plane of Maximum Symmetry.

Now the process described in my paper in the *Philosophical Magazine* leads to three mutual rectangular planes defined by Σ_1^2 and by Σ_2^2, and Σ_3^2 the other two roots of the cubic. Σ_2^2 and Σ_3^2 are easily found from a quadratic equation and are maxima values of the Mean Square Deviations. It might at first be supposed that these are the very planes we need. But we must remember that the first standard plane has been obtained on the basis of its approximation to the mirror plane, and only the mid-sagittal points have been used in its determination. It is accordingly somewhat one-sided to use solely these points in determining planes, which we need only limit as planes necessarily perpendicular to the Plane of Maximum Symmetry. Unfortunately when we come to the Horizontal Plane, and

* For example: Does asymmetry increase as we pass from more primitive to more highly civilised groups?

the Transverse Vertical Plane, we have no long series of points like the mid-sagittal points from which to determine approximate planes. The Apex and the Kappa are not natural points of the skull, but artificial points resulting from the definition of the Frankfurt Horizontal Plane. The Poria may be considered more nearly natural points although the Frankfurt Plane enters into their determination. The Orbitalia may be considered again as only in part natural points, as they depend upon the Poria, and the *lowest* points on the orbital rims are really meaningless, without the conception of the Horizontal Plane. But there is a greater difficulty arising here. When the skull is adjusted on the craniophor to the Frankfurt Horizontal Plane it will be found that in some cases several millimetres of the lower borders of the orbits may practically be parallel to that plane. This may not modify seriously the plans of the Orbitalia, but it renders their elevations above the drawing-board when the auricular axis of the skull is made perpendicular to the board, occasionally difficult of accurate determination.

Thus although it is perfectly easy to develop the mathematical solution for determining a plane perpendicular to a given plane, and fitting closely by a minimum square residual to a selected system of points—assumed in the case of a skull truly symmetrical to lie in that plane—yet in practice it is not easy to select such a system of points in the case either of the Horizontal Plane, or of the Transverse Vertical Plane.

I will not give here the mathematical theory by which a plane is constructed perpendicular to a given plane and of closest fit to a selected series of points, but merely indicate what resulted in the case of the Fuegian skull. The points selected were:

the two Poria (0, 0, 253·8) and (0, 0, 126·2),

and the two Orbitalia (13·3, 82·4, 219·0) and (5·1, 82·4, 145·8).

The equation to the plane perpendicular to the Plane of Maximum Symmetry of this skull and of closest fit to the above four points is*

$$\cdot 967{,}9125\,x - \cdot 250{,}8948\,y + \cdot 014{,}0420\,z + 13{\cdot}946{,}272 = 0.$$

This plane makes an angle of 14° 33′·2 with the plane of $x = 0$, and is accordingly not very close to the Frankfurt Horizontal Plane. Clearly the plane perpendicular to our First Standard Plane and fitting closely to the Poria and Orbitalia is not, for this skull at any rate, at all approximated to by the Frankfurt Horizontal Plane.

The above plane is the plane of Least Square Residual from the Poria and Orbitalia which is perpendicular to the First Standard Plane. The plane of maximum mean square deviation subject also to the perpendicularity condition is

$$\cdot 251{,}0725\,x + \cdot 967{,}8841\,y - \cdot 012{,}7715\,z - 15{\cdot}889{,}862 = 0.$$

This corresponds to the Transverse Vertical Plane, but it makes an angle of 14° 33′·6 with it.

* The plane determined in this way passes through the centroid of the Poria and Orbitalia.

These results are not satisfactory, if we wish our second and third standard planes to approximate fairly closely to the Frankfurt Horizontal Plane and the customary Transverse Vertical Plane. Accordingly, it seemed worth while investigating whether the other two roots of the fundamental cubic would give a system of rectangular planes with adequate approximation to the Frankfurt Horizontal Plane and to the customary Transverse Vertical Plane.

In the case of the Fuegian skull the three roots of the cubic are

$$\Sigma_1^2 = 1{\cdot}676{,}8435,$$
$$\Sigma_2^2 = 5121{\cdot}492{,}8812,$$
$$\Sigma_3^2 = 2848{\cdot}293{,}3502.$$

Σ_2^2 with the greater maximal value of the Mean Square Residual led to the plane

$$\cdot 4312{,}1286\,x + \cdot 9021{,}9390\,y - \cdot 0100{,}8176\,z = 12{\cdot}8930{,}3076.$$

This plane is closest to the Transverse Vertical Plane, or the reference plane $y = 0$, but it makes an angle of 25° 33′·1 with it.

The other maximum Σ_3^2 provides the plane

$$-\cdot 9021{,}9942\,x + \cdot 4310{,}4409\,y - \cdot 0154{,}0415\,z = 22{\cdot}6879{,}0655,$$

which corresponds to the Frankfurt Horizontal Plane, but makes an angle with it of 25° 33′·1, equal to that which the previous plane makes with the Transverse Vertical Plane. This did not seem very hopeful, but the like planes were worked out for the Negro skull from the Teita Hills. The three roots of the fundamental cubic (given on p. 237) are

$$\Sigma_1^2 = 1{\cdot}1451{,}0147, \quad \Sigma_2^2 = 4918{\cdot}7879{,}4908, \quad \Sigma_3^2 = 2430{\cdot}9817{,}4235.$$

Σ_2^2 gives us the equation

$$\cdot 0872{,}1082\,x + \cdot 9961{,}3765\,y + \cdot 0102{,}0028\,z = 9{\cdot}548{,}7522.$$

This plane is closest to the Transverse Vertical Plane, or $y = 0$; it makes an angle with it of 5° 3′·2.

Σ_3^2 gives us

$$-\cdot 9959{,}4649\,x + \cdot 0874{,}1127\,y - \cdot 0212{,}0966\,z = 9{\cdot}397{,}5540,$$

which makes an angle of $-5°\ 9'{\cdot}6$ with the plane of $x = 0$, or the Frankfurt Horizontal Plane. These results for the Negro skull are better than those for the Fuegian, but are not satisfactory, if we desire our Second and Third Standard Planes to approach fairly closely to the Frankfurt Horizontal Plane and the customary Transverse Vertical Plane.

No doubt the three roots of the fundamental cubic would provide three standard mutually rectangular planes possessing certain very definite physical relations with regard to any skull, but they would fail to provide close approximations to the Frankfurt Horizontal Plane. Accordingly, I started on a different route to find my Second Standard Plane. I sought a plane which should be at right angles to the First Standard Plane, i.e. the Plane of Maximum Symmetry, and should have a maximum cosine, i.e. a minimum angle with the Frankfurt Horizontal Plane. This should

form the Second Standard Plane. The Third Standard Plane must be perpendicular to the First and Second Standard Planes, and this will fully determine its direction cosines.

Mathematically: let λ, μ, ν be the direction cosines of the Second Standard Plane; l, m, n those of the First Standard Plane, and λ', μ', ν' of the Third Standard Plane. Let L, M, N be the direction cosines of the plane with which λ, μ, ν is to make a minimum angle θ or a maximum cosine $u = \cos\theta$. Then

$$u = L\lambda + M\mu + N\nu,$$
$$0 = l\lambda + m\mu + n\nu,$$
$$\lambda^2 + \mu^2 + \nu^2 = 1.$$

Hence for a maximum of u, if A and B be indeterminate multipliers

$$l + AL + B\lambda = 0, \quad m + AM + B\mu = 0, \quad n + AN + B\nu = 0.$$

Whence $$1 + A(Ll + Mm + Nn) = 0, \quad Au_0 + B = 0.$$

Accordingly,
$$L(1 - l^2) - Mlm - Nnl = \lambda u_0$$
$$-Lml + M(1 - m^2) - Nmn = \mu u_0$$
$$-Lnl - Mnm + N(1 - n^2) = \nu u_0.$$

Whence

$$\frac{\lambda}{L(1-l^2) - Mlm - Nln} = \frac{\mu}{-Lml + M(1-m^2) - Nmn}$$
$$= \frac{\nu}{-Lnl - Mnm + N(1-n^2)} = \frac{1}{u_0} \quad \text{.....(xiii),}$$

which solve the general problem.

But for our special case: $L = 1$, $M = 0$, $N = 0$, and thus

$$\frac{\lambda}{1 - l^2} = \frac{\mu}{-ml} = \frac{\nu}{-nl} = \frac{1}{u_0} = \frac{1}{\sqrt{1 - l^2}}.$$

Thus $$\lambda = \sqrt{1 - l^2}, \quad \mu = -\frac{lm}{\sqrt{1 - l^2}}, \quad \nu = -\frac{ln}{\sqrt{1 - l^2}} \quad \text{.....(xiv),}$$

these determine the direction cosines of the Second Standard Plane.

For the Third Standard Plane we have

$$l\lambda' + m\mu' + n\nu' = 0$$

and
$$\lambda\lambda' + \mu\mu' + \nu\nu' = 0,$$

or
$$\frac{\lambda'}{m\nu - n\mu} = \frac{\mu'}{n\lambda - l\nu} = \frac{\nu'}{l\mu - m\lambda}.$$

Or again substituting for λ, μ, ν,

$$\frac{\lambda'}{0} = \frac{\mu'}{n\sqrt{1 - l^2} + \dfrac{l^2 n}{\sqrt{1 - l^2}}} = \frac{\nu'}{\dfrac{-l^2 m}{\sqrt{1 - l^2}} - m\sqrt{1 - l^2}}.$$

Thus λ' must = 0, and

$$\frac{\mu'}{n} = -\frac{\nu'}{m} = \frac{1}{\sqrt{1 - l^2}},$$

Accordingly, we have finally:

$$\lambda' = 0, \quad \mu' = \frac{n}{\sqrt{1-l^2}}, \quad \nu' = -\frac{m}{\sqrt{1-l^2}} \quad \text{.....(xv).}$$

We have thus reached on our hypothesis fully determined directions of the three standard planes.

The reader will observe that our Third Standard Plane is invariably at right angles to the plane of $x = 0$, since $\lambda' = 0$, that is to the Frankfurt Horizontal Plane. We may now test how satisfactorily this arrangement works on our six illustrative crania.

(*A*) *Fuegian Skull.*

$$l = -\cdot0095{,}7040, \quad m = +\cdot0157{,}3352, \quad n = +\cdot9998{,}3036.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot9999{,}5420, \quad \mu = +\cdot0001{,}5060, \quad \nu = +\cdot0095{,}6921.$$

The Second Standard Plane makes an angle of $0^\circ\ 32'\cdot9$ with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot9998{,}7615, \quad \nu' = -\cdot015{,}3424.$$

The Third Standard Plane makes an angle of $+0^\circ\ 54'\cdot1$ with the customary Transverse Vertical Plane.

(*B*) *Ancient Egyptian Skull from Nubia.*

$$l = +\cdot005{,}4027, \quad m = -\cdot006{,}2292, \quad n = +\cdot999{,}9660.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot9999{,}8540, \quad \mu = +\cdot0000{,}3365, \quad \nu = -\cdot0054{,}0260.$$

The Second Standard Plane makes an angle of $0^\circ\ 11'\cdot6$ with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot9999{,}8060, \quad \nu' = +\cdot0062{,}2929.$$

The Third Standard Plane makes an angle of $+\ 0^\circ\ 21'\cdot4$ with the customary Transverse Vertical Plane.

(*C*) *Arab Skull.*

$$l = -\cdot0040{,}6975, \quad m = -\cdot0423{,}9970, \quad n = +\cdot9990{,}9244.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot9999{,}9127, \quad \mu = -\cdot0001{,}7256, \quad \nu = +\cdot0040{,}6609.$$

The Second Standard Plane makes an angle of $0^\circ\ 14'\cdot3$ with the Frankfurt Horizontal Plane.

Direction Cosines of the Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot9991{,}0116, \quad \nu' = +\cdot0424{,}0007.$$

The Third Standard Plane makes an angle of $2^\circ\ 25'\cdot8$ with the customary Transverse Vertical Plane.

(*D*) *Teita Negro Skull.*

$$l = -\cdot 0220{,}1936, \quad m = -\cdot 0083{,}0922, \quad n = +\cdot 9997{,}2301.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot 9997{,}5754, \quad \mu = -\cdot 0001{,}8301, \quad \nu = -\cdot 0220{,}1860.$$

The Second Standard Plane makes an angle of 1° 15′·7 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot 9999{,}6546, \quad \nu' = +\cdot 0083{,}1123^5.$$

The Third Standard Plane makes an angle of + 0° 28′·5 with the customary Transverse Vertical Plane.

(*E*) 17*th Century English Skull.*

$$l = +\cdot 0081{,}2136, \quad m = -\cdot 0109{,}6533, \quad n = +\cdot 9999{,}0690.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot 9999{,}6702, \quad \mu = +\cdot 0000{,}8906, \quad \nu = -\cdot 0081{,}2087.$$

The Second Standard Plane makes an angle of 0° 27′·9 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot 9999{,}3988, \quad \nu' = +\cdot 0109{,}6569.$$

The Third Standard Plane makes an angle of 0° 37′·7 with the customary Transverse Vertical Plane.

(*F*) *Hindu Skull.*

$$l = -\cdot 0017{,}4154, \quad m = -\cdot 0054{,}1230, \quad n = +\cdot 9999{,}8384.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot 9999{,}9848, \quad \mu = -\cdot 0000{,}0943, \quad \nu = +\cdot 0017{,}4151.$$

The Second Standard Plane makes an angle of 0° 6′·0 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot 9999{,}8536, \quad \nu' = +\cdot 0054{,}1231.$$

The Third Standard Plane makes an angle of 0° 18′·6 with the customary Transverse Vertical Plane.

As a result we see that our Second Standard Plane as defined above makes a mean angle (28′·1) for the six skulls with the Frankfurt Horizontal Plane of less than half a degree, while the Third Standard Plane makes a mean angle (51′·0), less than a degree, with the customary Transverse Vertical Plane. Thus with the above definitions of the Second and Third Standard Planes we reach planes which make respectively only small angles with such very familiar planes as the Frankfurt Horizontal Plane and the Transverse Vertical Plane.

One point alone remains unsettled, namely: We have determined by (xiv) and (xv) the direction cosines of these Standard Planes as functions solely of the direction cosines of the Plane of Maximum Symmetry. But we have not determined the meet of our three Standard Planes. This remains to be selected. After dealing with several points I came to the conclusion that the most suitable point to choose as origin of the Standard Planes was the point in which the First Standard Plane or Plane of Maximum Symmetry meets the auricular axis. The elevations of these points are respectively Fuegian, 189·054,163; Egyptian from Nubia, 188·671,058; Arab, 193·357,334; Teita Negro, 186·435,955; English, 196·758,098; Hindu, 186·608,610.

Paying attention to which Porion was uppermost (as noted on pp. 236—238) we find that in no case is the intersection more than 1·5 mm. from the Mid-porion, the maximum being 1·4427 in the case of the Arab skull. Four intersections deviate from the Mid-porion towards the Left Porion and two towards the Right Porion, the average of the deviations of the six intersections is only ·0136 mm. towards the Right Porion. The point chosen seems therefore a reasonable one, and enables us to write down very readily the equations to the Standard Planes. For our six skulls they are as follows:

(*A*) *Fuegian Skull.*

1st Standard Plane:

$$-\cdot009{,}5704\,x+\cdot015{,}7335\,y+\cdot999{,}8304\,z=189\cdot022{,}082,$$

2nd Standard Plane:

$$\cdot999{,}9542\,x+\cdot000{,}1506\,y+\cdot009{,}5692\,z=1\cdot809{,}0990,$$

3rd Standard Plane:

$$-\cdot999{,}8762\,y+\cdot015{,}3424\,z=2\cdot900{,}5446.$$

(*B*) *Egyptian Skull* (from Nubia).

1st Standard Plane:

$$\cdot005{,}4027\,x-\cdot006{,}2292\,y+\cdot999{,}9660\,z=188\cdot664{,}644,$$

2nd Standard Plane:

$$-\cdot999{,}9854\,x-\cdot000{,}0337\,y+\cdot005{,}4026\,z=1\cdot019{,}3143,$$

3rd Standard Plane:

$$\cdot999{,}9806\,y+\cdot006{,}2293\,z=1\cdot175{,}2867.$$

(*C*) *Arab Skull.*

1st Standard Plane:

$$-\cdot004{,}0698\,x-\cdot042{,}3997\,y+\cdot999{,}0924\,z=193\cdot181{,}851,$$

2nd Standard Plane:

$$\cdot999{,}9913\,x-\cdot000{,}1726\,y+\cdot004{,}0661\,z=\cdot786{,}2103,$$

3rd Standard Plane:

$$\cdot999{,}1012\,y+\cdot042{,}4001\,z=8\cdot198{,}3145.$$

(*D*) *Teita Negro Skull.*

1st Standard Plane:

$$-\cdot 022{,}0194\,x - \cdot 008{,}3092\,y + \cdot 999{,}7230\,z = 186\cdot 384{,}314,$$

2nd Standard Plane:

$$-\cdot 999{,}7575\,x + \cdot 000{,}1830\,y + \cdot 022{,}0186\,z = 4\cdot 105{,}0587,$$

3rd Standard Plane:

$$\cdot 999{,}9655\,y + \cdot 008{,}3112\,z = 1\cdot 549{,}5065.$$

(*E*) *English Skull.*

1st Standard Plane:

$$\cdot 008{,}1214\,x - \cdot 010{,}9653\,y + \cdot 999{,}9069\,z = 196\cdot 739{,}780,$$

2nd Standard Plane:

$$-\cdot 999{,}9670\,x - \cdot 000{,}0891\,y + \cdot 008{,}1209\,z = 1\cdot 597{,}8469,$$

3rd Standard Plane:

$$\cdot 999{,}9399\,y + \cdot 010{,}9657\,z = 2\cdot 157{,}5883.$$

(*F*) *Hindu Skull.*

1st Standard Plane:

$$-\cdot 001{,}7415\,x - \cdot 005{,}4123\,y + \cdot 999{,}9838\,z = 186\cdot 605{,}594,$$

2nd Standard Plane:

$$\cdot 999{,}9985\,x - \cdot 000{,}0094\,y + \cdot 001{,}7415\,z = \cdot 324{,}9808,$$

3rd Standard Plane:

$$\cdot 999{,}9854\,y + \cdot 005{,}4123\,z = 1\cdot 009{,}9836.$$

The reader must bear in mind what these three Standard Planes signify:

The First Standard Plane is the Plane of Maximum Symmetry, or our nearest approach to a mirror plane, the plane which deviates least from the mid-sagittal points as judged by the Mean Square Residual.

The Second Standard Plane passes through the point where the first meets the auricular axis, is perpendicular to the first and makes the minimum angle with the Frankfurt Horizontal Plane, which fails in the condition of being perpendicular to the First Standard Plane. Thus our Second Standard Plane may be looked upon as an improved "Horizontal Plane."

The Third Standard Plane passes through the point where the first two meet the auricular axis, and is perpendicular to both of them. It may therefore be looked upon as an improved Transverse Vertical Plane.

The Second and Third Standard Planes will not as a rule pass through the auricular axis and neither will generally contain the Poria. The second will not usually pass through the Kappa, nor the third through the Apex. It may be not without interest to measure the mean departures of the Second Plane for our six skulls from the Poria and the Kappa, and of our Third Plane from the Poria and the Apex *.

* In the equations to the planes, as on pp. 236—238, it is needful to go to 6 or 7 decimals in order to get values of the direction cosines which will provide angles to a decimal of a minute. The values, being worked from those equations, are written down to the same number of decimals, but to use two or three decimal places in the distances is of course ample.

(*A*) *Fuegian Skull.*

2nd, or Horizontal Standard Plane.

The κ is above this plane with the perpendicular upon it = − ·000,4302. The R. Porion is above the plane − ·619,5640 and the L. Porion below it + ·601,4660. Thus, as judged by the Poria, the right ear is about ·6 above, and the left ear about ·6 mm. below the Standard Horizontal Plane.

3rd, or Transverse Vertical Standard Plane.

The Apex is + ·008,5022 in front of this plane which like the deviation of the κ is not sensible on the skull. The R. Porion is behind this Transverse Vertical Plane − ·993,3565 and the L. Porion in front of it + ·964,3337. Thus the left ear comes horizontally forwards, and the right ear retreats.

(*B*) *Nubian Egyptian Skull.*

2nd, or Horizontal Plane.

The κ is above this plane at − ·000,0522 again an insensible difference. The R. Porion is above it at − ·315,1279, and the L. Porion is below it at + ·302,3893. Again the right ear is raised above the Horizontal Plane, and the left lowered below it although the differences are only about ·3 mm.

3rd, or Transverse Vertical Standard Plane.

The Apex is at − ·006,4115 from this plane; the R. Porion is − ·363,3504, i.e. behind, and the L. Porion about + ·348,6586, i.e. in front of it, or the Poria are about one-third of the distance from this plane of the Poria in the case of the Fuegian skull.

(*C*) *Arab Skull.*

2nd, or Horizontal Standard Plane.

The κ is here + ·002,1754, practically at an insensible distance below the Horizontal Plane. The R. Porion is below at + ·241,7595 and the L. Porion is above the Horizontal Plane at − ·253,4915.

3rd, or Transverse Vertical Standard Plane.

The Apex is behind this plane at a distance − ·116,3451. The R. Porion is + 2·520,9411, i.e. behind, and the L. Porion − 2·643,3911, i.e. in front of the Transverse Vertical Plane. These are the most considerable deviations we have found for any of the six skulls.

(*D*) *Teita Negro Skull.*

2nd, or Horizontal Standard Plane.

The κ here is at a distance + ·047,0995 below this plane. The R. Porion is + 1·229,4295, i.e. below the Standard Horizontal, and the L. Porion is − 1·219,0388, i.e. above it.

3rd, or Transverse Vertical Standard Plane.

The Apex is in front of this plane at a distance + ·017,7523. The R. Porion is behind the Standard Transverse Vertical at + ·464,0638, and the L. Porion in front of it at − ·460,1417.

(*E*) *English 17th Century Skull.*

2nd, or Horizontal Standard Plane.

The κ is here $-\cdot017,8341$ above the Horizontal Plane. The R. Porion is at $-\cdot503,0299$, i.e. above, and the L. Porion at $+\cdot515,3309$, i.e. below it.

3rd, or Transverse Vertical Standard Plane.

The Apex is behind this plane at a distance of $-\cdot032,2620$. The R. Porion is at $-\cdot679,2383$, i.e. in front, and the L. Porion at $+\cdot695,8605$, i.e. behind the Transverse Vertical Plane.

(*F*) *Hindu Skull.*

2nd, or Horizontal Standard Plane.

The κ is $+\cdot001,3340$ below this plane. The R. Porion is at $+\cdot097,8892$, i.e. below, and the L. Porion at $-\cdot095,7656$, i.e. above the Horizontal Plane.

3rd, or Transverse Vertical Standard Plane.

The Apex is behind this plane at $-\cdot004,2814$ distance. The R. Porion is at $+\cdot304,2197$, i.e. in front, and the L. Porion is at $-\cdot297,6281$, i.e. behind the Transverse Vertical Plane.

Reviewing these results as a whole, we conclude as follows:

Our Standard Horizontal and Transverse Vertical Planes, which we believe to be more reasonable than the usual Frankfurt Horizontal and Transverse Planes because they are at right angles to the Plane of Maximum Symmetry, do not pass through the auricular axis, but the first passes very nearly through the Kappa, the average distance in the cases of our six skulls from it being only ·0115 mm. regardless of sign. We have previously seen that our Standard Horizontal Plane passes, with a mean deviation of ·0136 mm. only, close to the Mid-porion (p. 248). The Frankfurt Horizontal Plane also passes through these two points. If now the Frankfurt Horizontal Plane be turned round the line joining these two points as axis, it will pass to the position of the Standard Horizontal Plane, one Porion approaching it and one receding from it. This provides the difference in vertical height of the two Poria, or speaking popularly of the two ears.

The Second Standard Plane passes just as closely to the Mid-porion and very close to the Apex, i.e. at an average distance of ·0309 mm. from it. It is also perpendicular to the Frankfurt Plane. Accordingly if the usual Transverse Vertical Plane be rotated about the line joining Apex to Mid-porion through a small angle it will come nearly into the position of our Standard Transverse Vertical Plane. The rotation is, to judge from our six skulls only, somewhat greater than the rotation required to change the Frankfurt Horizontal into the Standard Horizontal Plane. The result of this rotation is to cause one Porion to retreat behind the Standard Transverse Plane and the other to advance in front of it. Popularly, one ear may be said with reference to the Plane of Maximum Symmetry to be farther back on the head than the other ear.

This more detailed numerical investigation indicates how the ears are displaced vertically and horizontally with regard to homologous positions as judged from a good representative of the mirror plane. Of course this solely emphasises in a different manner the point we had already reached, namely that the auricular axis is not perpendicular to the Plane of Maximum Symmetry.

The auricular axis of the skull is invaluable as an aid to the determination of cranial planes of reference, but it is far from an essential base line when we are investigating the symmetry of the skull. It cannot be legitimately used in determining standard horizontal and vertical planes, for in doing so we are assuming that it is necessarily perpendicular to a well-chosen Median Sagittal Plane, and this it is not.

Summary.

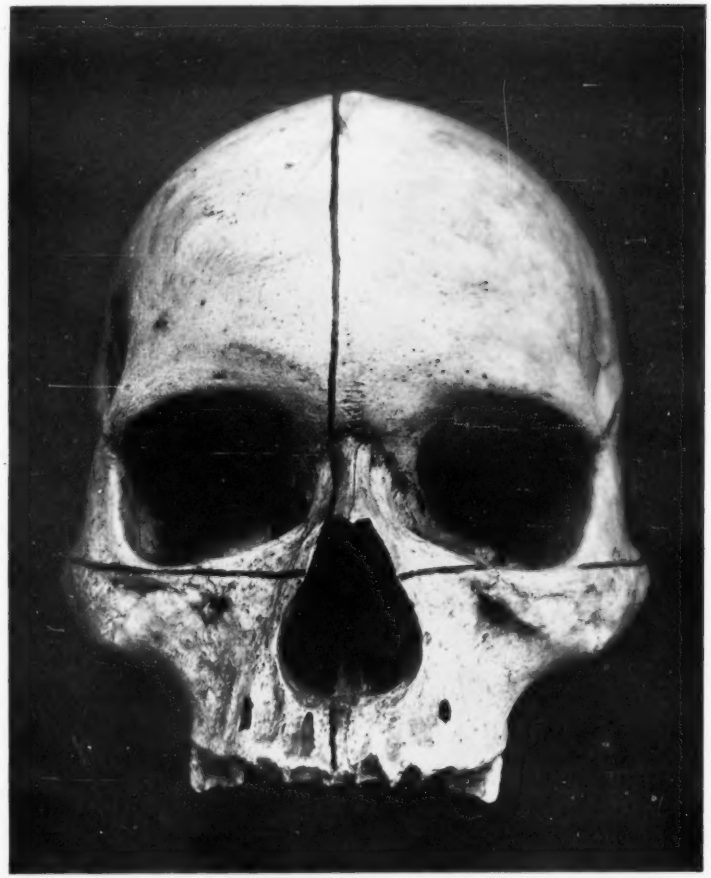
By aid of the Cranial Coordinatograph it has been possible to obtain the coordinates of any point on the skull referred to three rectangular planes of reference. Thence we obtain the analytical equations of solid geometry to any lines or planes of the skull. Discussing the "mirror plane" of an absolutely symmetrical skull, we were led to find as our First Standard Plane the plane of the Mininum Square Deviation from the "mid-sagittal" points. This plane is that of maximum mesial symmetry. The Second and Third Standard Planes of the skull must be at right angles to each other and to the Plane of Maximum Symmetry. To make as little change as feasible we took our Second Standard Plane to make a minimum angle with the Frankfurt Horizontal Plane. This second plane practically passes through the Kappa and the Mid-porion, but is tilted to the Frankfurt Horizontal Plane. It is a truer Standard Horizontal Plane than the Frankfurt in that it is perpendicular to the close fitting Mid-sagittal Plane. Our Third Standard Plane is perpendicular to the first two and is a truer Transverse Vertical Plane than that through the auricular axis. These two standard planes indicate well the actual shift of the ear-orifices forwards and backwards, upwards and downwards, being, as we might naturally anticipate, an effect of the general asymmetry of the skull. The ears, being at a maximum distance from the Mid-sagittal Plane, show more than any other part of the skull, or of the living head, a maximum displacement from a symmetrical position, and really preclude the use of the auricular axis for the determination of a true horizontal, or a true transverse vertical plane.

The reader may ask: How much labour is involved in reducing a skull to analytical solid geometry? Originally it took me about two hours to two hours and a half to determine the Poria, Kappa and Apex on a Ranke craniophor, to adjust the skull on the staddle and read off the coordinates of 15 to 20 points. But with practice I think two skulls could easily be done in a morning of less than three hours. The numerical work is very straightforward computing, but laborious. The solving of the cubic is not so lengthy as it may appear, as we can be fairly sure that Σ_1^2 will lie between 1 and 3 sq. mm. Because its value is so small and the other roots of the cubic so large it is needful to keep a considerable number of decimal

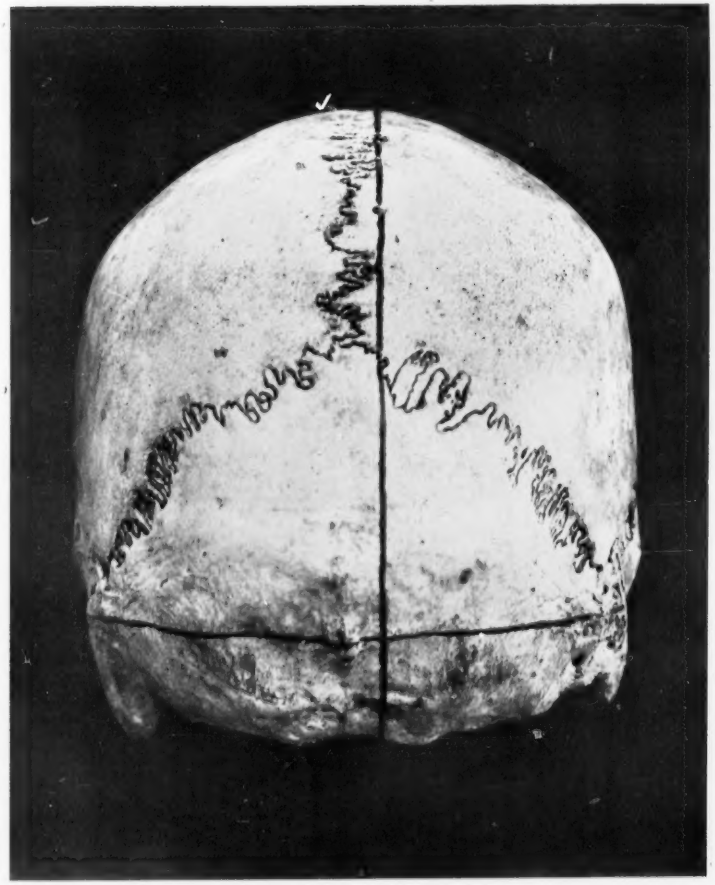
Plate I

Biometrika, Vol. XXV, Parts III and IV

Pearson: *The Cranial Coordinatograph and the Standard Planes of the Skull*



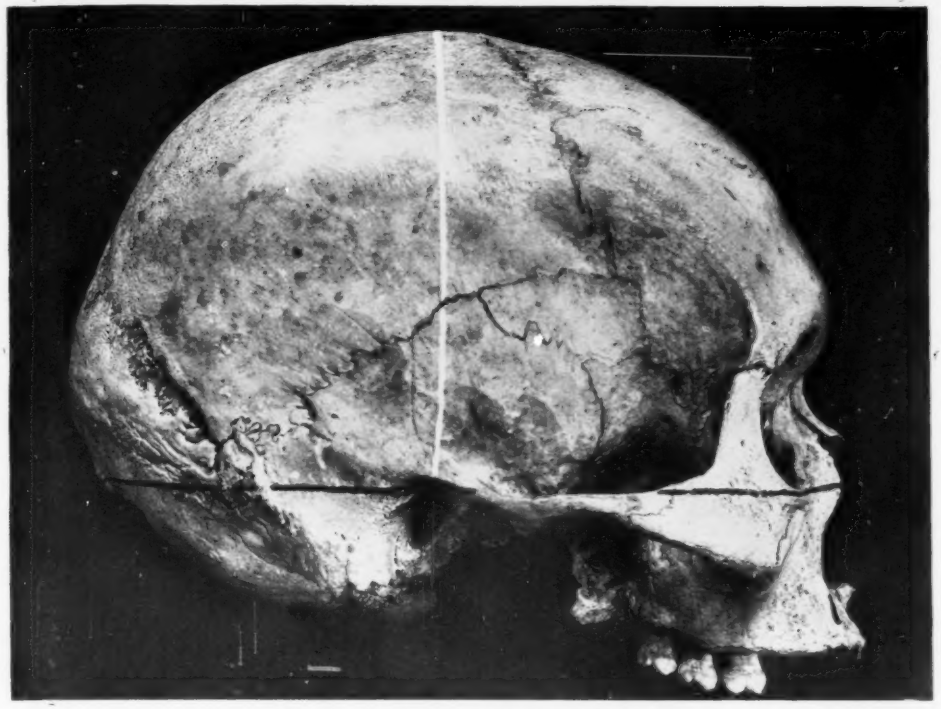
(*a*) Hindu Skull. *Norma facialis.*



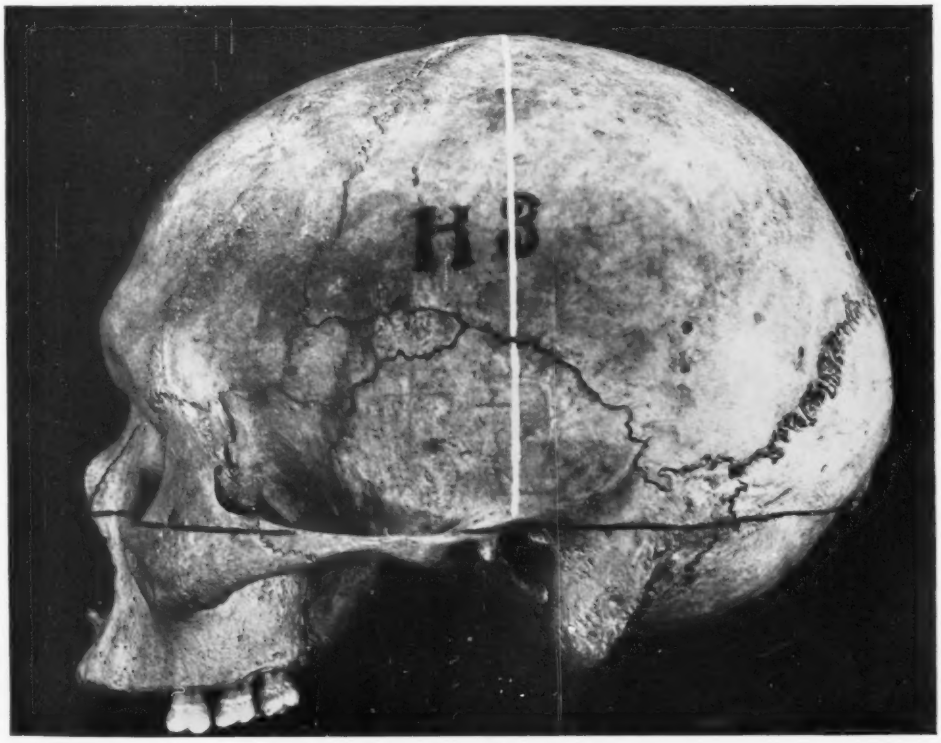
(*b*) Hindu Skull. *Norma occipitalis.*

This plate indicates how a plane perpendicular to the auricular axis and passing through the Mid-porion, and therefore perpendicular to the Frankfurt Horizontal Plane, fails to pass through any one of the "mid-sagittal points." The horizontal black curve is the trace of the Frankfurt Horizontal, and the vertical line is the trace of the plane through the Mid-porion perpendicular to the auricular axis.

UNIV. OF TORONTO



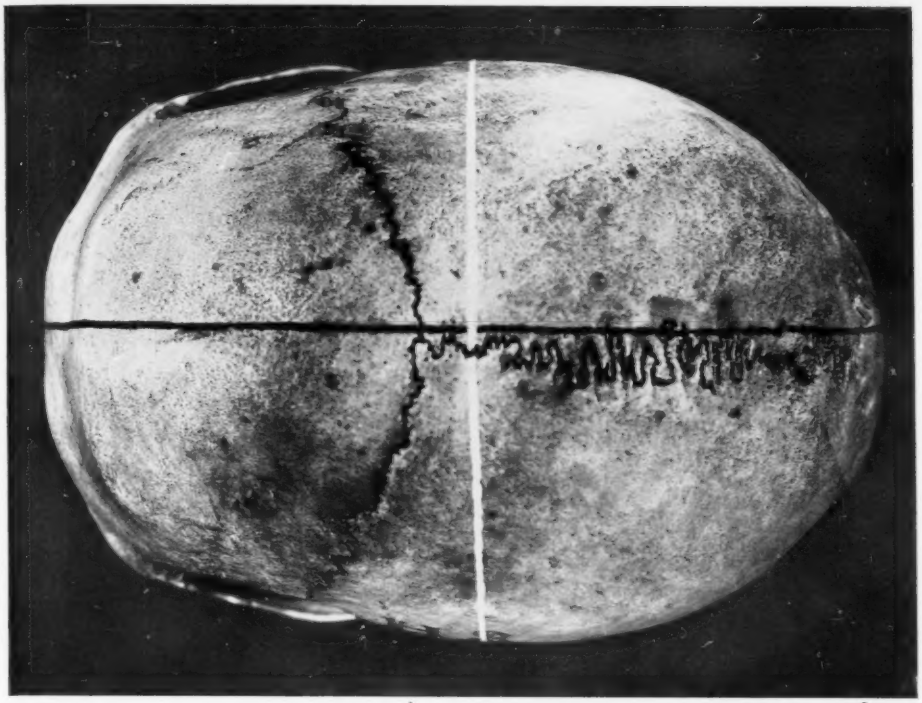
(*c*) Hindu Skull. *Norma lateralis* (Right Profile).



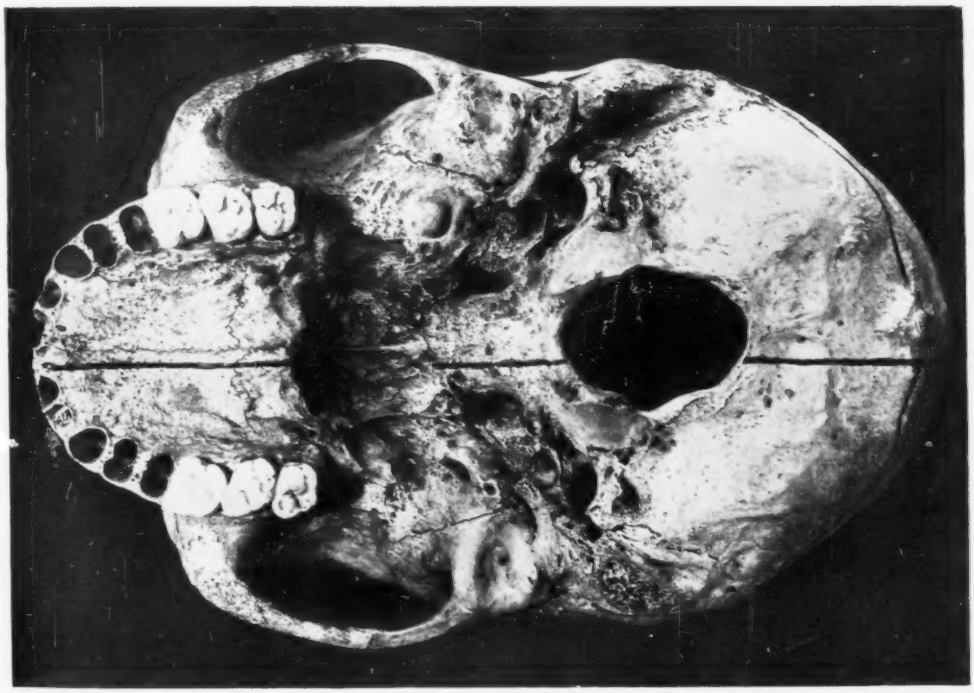
(*d*) Hindu Skull. *Norma lateralis* (Left Profile).

The white line is the trace on this skull of the customary Transverse Vertical Plane; the horizontal black line is the trace of the Frankfurt Horizontal Plane.

UNIV. OF MICH.



(*e*) Hindu Skull. *Norma verticalis.*



(*f*) Hindu Skull. *Norma basalis.*

The white line is the trace of the customary Transverse Vertical Plane; the horizontal black line is the trace of the plane through the Mid-porion perpendicular to the auricular axis.

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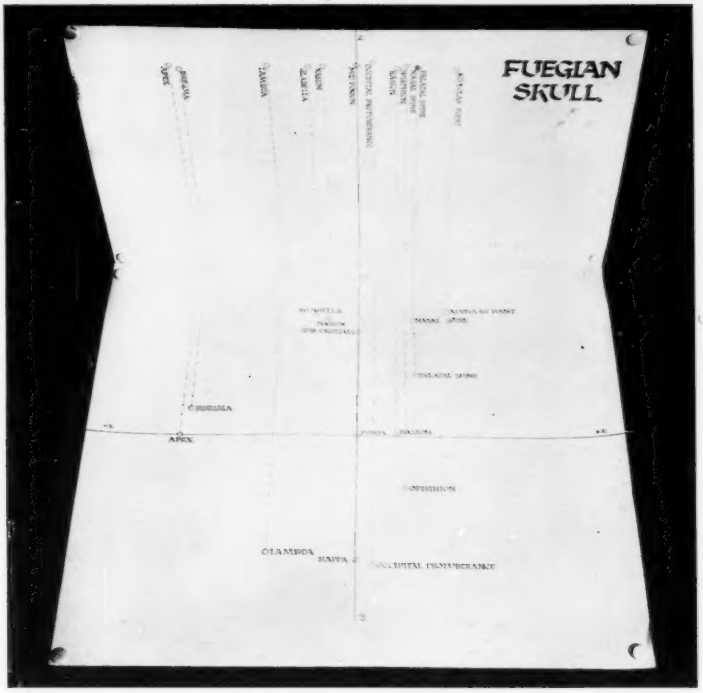
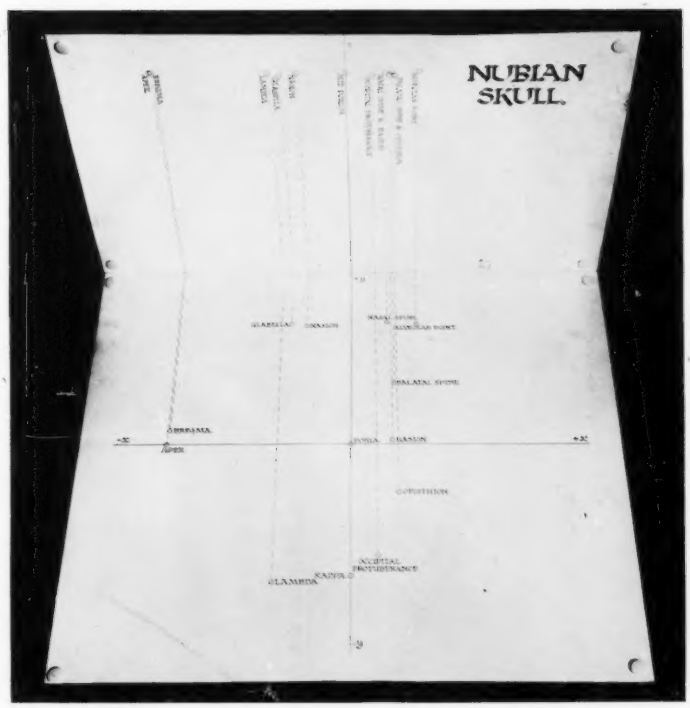
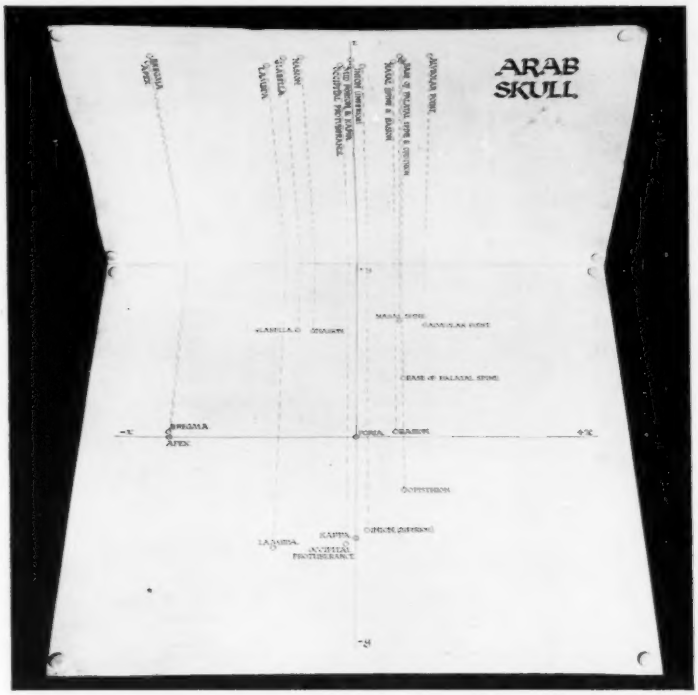


Plan and Elevation Model of a Fuegian Skull.



Plan and Elevation Model of an Egyptian Skull from Nubia.

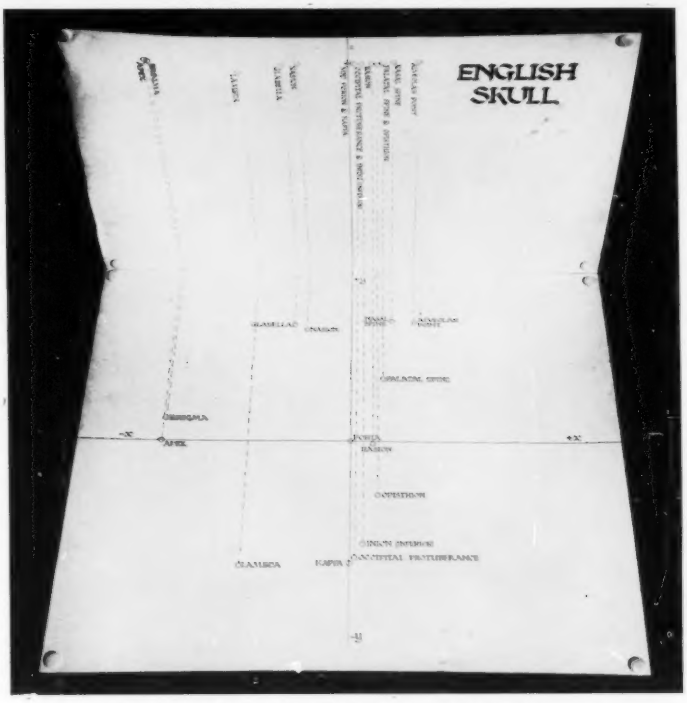
UNIV. OF ILLINOIS



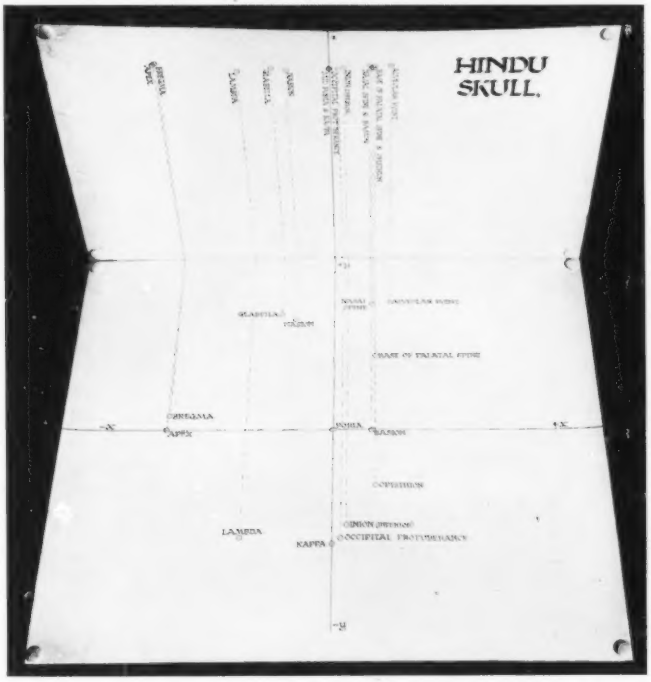
Plan and Elevation Model of an Arab Skull.

Plan and Elevation Model of a Negro Skull.

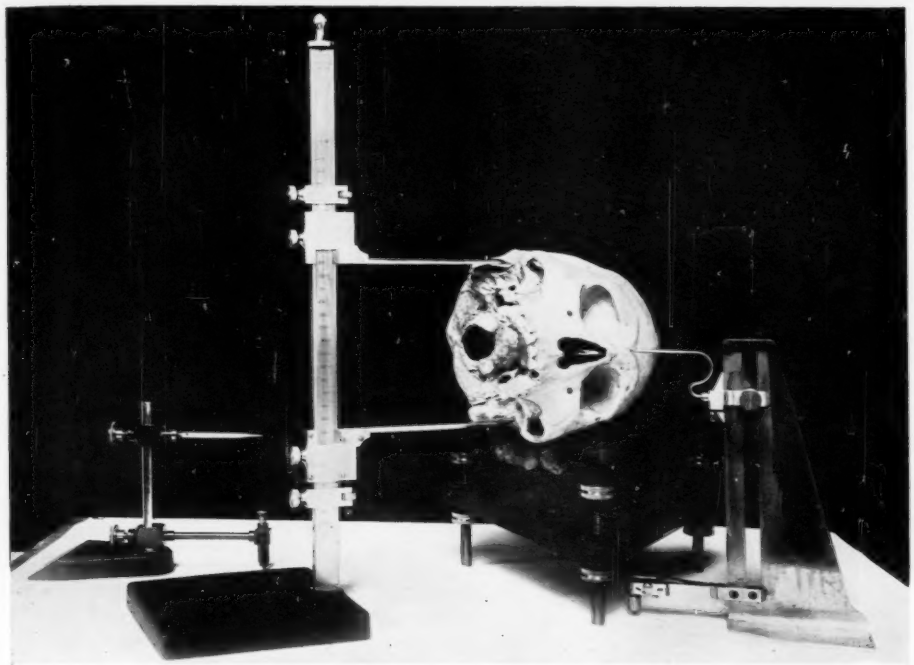
UNIV. OF MICH.



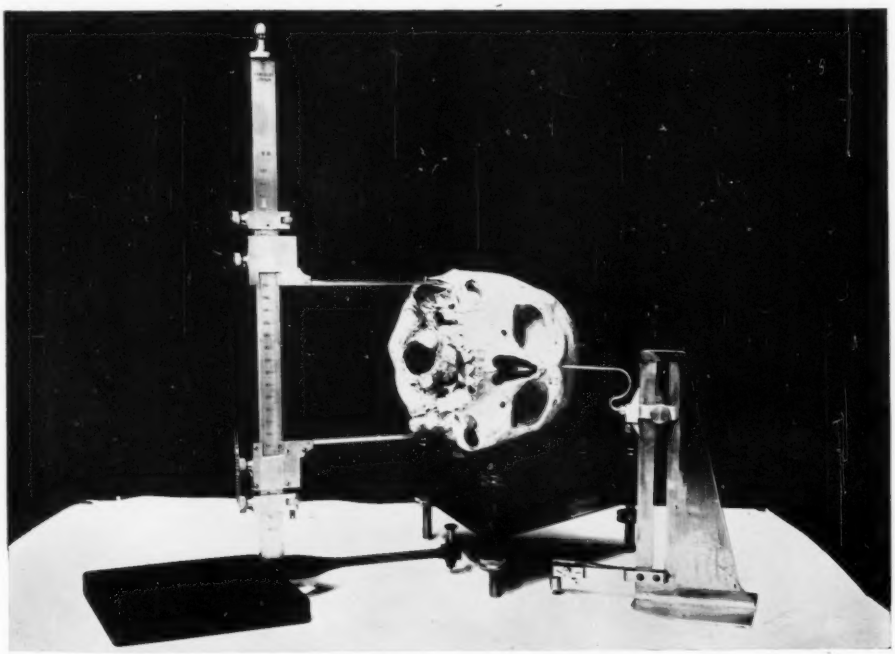
Plan and Elevation Model of an English Skull.



Plan and Elevation Model of a Hindu Skull.



(*a*) First form of Cranial Coordinatograph with Skull on Skull Staddle and auricular axis vertical. Klaatsch and Pearson independent projectors, the former set at glabella.



(*b*) Final form of Cranial Coordinatograph with a Pearson projector attached to its base, so that the plan is marked and elevation taken by the single instrument.

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(*a*)

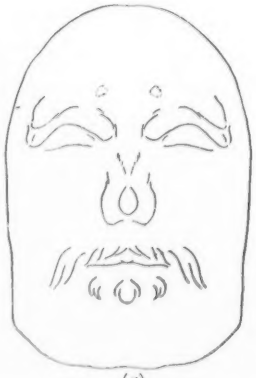
(i) Symmetricised Portrait of Right Side.

(*b*)

Drawing of Oliver Cromwell, after the bust in the Ashmolean Museum.

(*c*)

Symmetricised Portrait of Left Side.

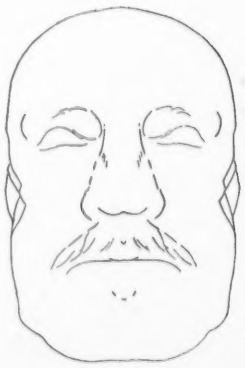


(*a*)

(ii) Symmetricised Drawing of Right Side.

(*b*)

Drawing of Cromwell's Death Mask.



(*c*)

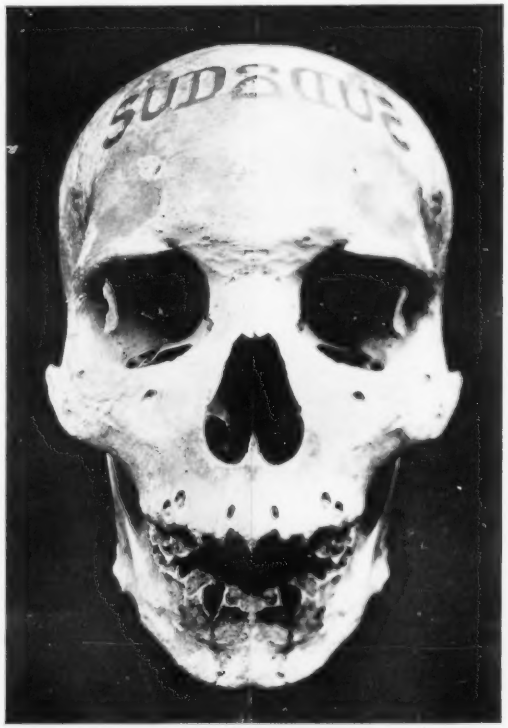
Symmetricised Drawing of Left Side.

(*a*)
Symmetricised Portrait.
Mirror Portrait of Right Side.

(*b*)
Full Face Natural Portrait.

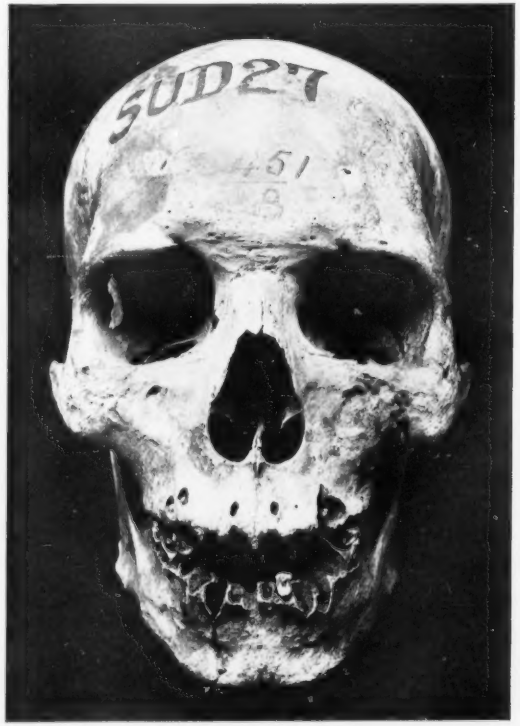
(*c*)
Symmetricised Portrait.
Mirror Portrait of Left Side.

This plate indicates how much of personal character depends on asymmetry of the face.



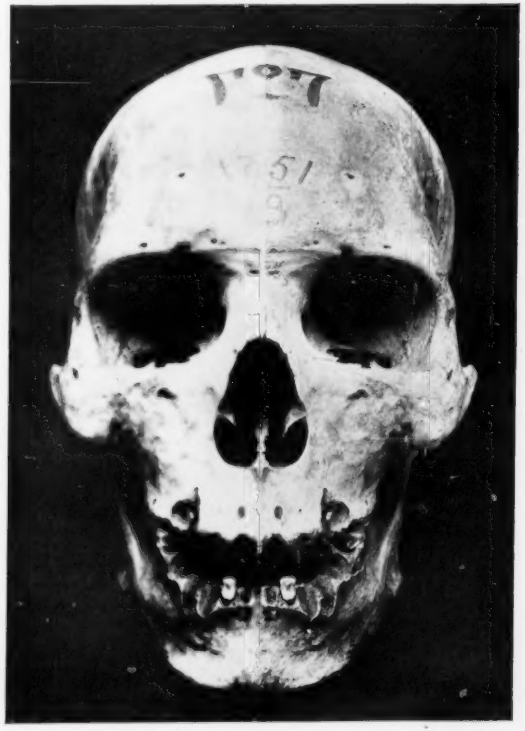
(*a*)

Symmetricised *Norma facialis*.
Mirror Portrait of Right Side.



(*b*)

Natural *Norma facialis* of Ancient Egyptian Skull
from Nubia.



(*c*)

Symmetricised *Norma facialis*.
Mirror Portrait of Left Side.

places. I think that two skulls could be dealt with in 5 to 6 hours by a good computer, so that an investigator could make a daily output of two skulls, or 10 to 12 a week. A month to five weeks would complete a racial sample of some 50 crania. Thus 10 to 12 races could be studied in a year to eighteen months. This is less time than many students give to their thesis for a doctorate, and a most valuable study of the standard planes of the skull and its asymmetry would result.

The present paper only professes to be illustrative of what may be achieved by the use of a coordinatograph, and the application of solid geometry to craniometry.

I may be over-enthusiastic, but I unhesitatingly believe that there is a most promising field for the craniometricians who are the first to apply Cartesian geometry to the skull.

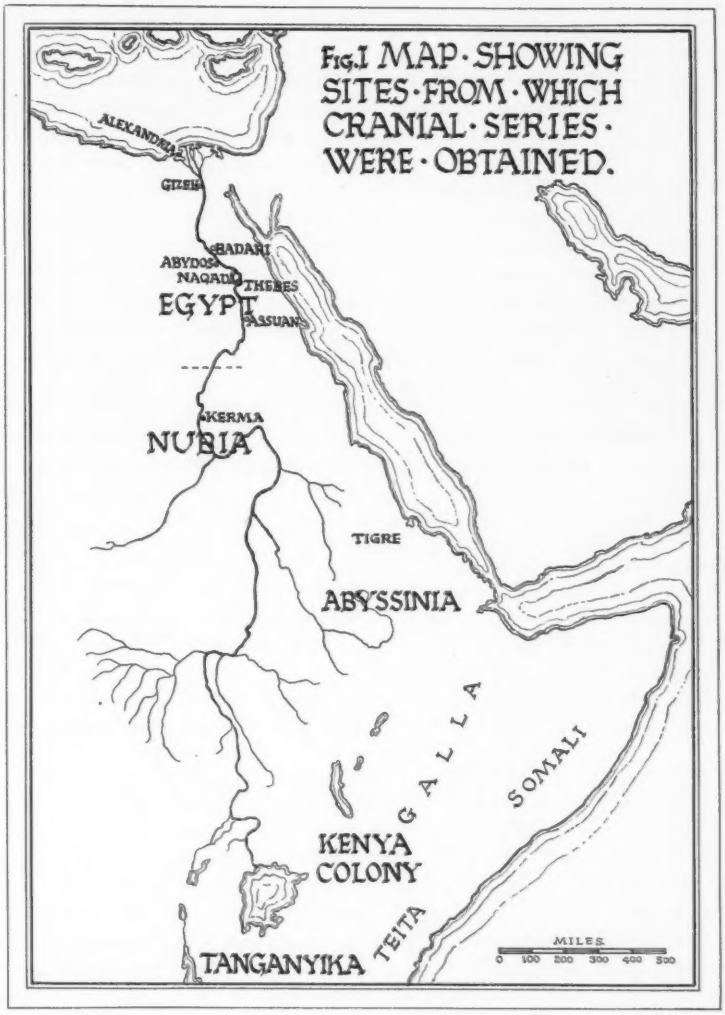
A STUDY OF TWELFTH AND THIRTEENTH DYNASTY SKULLS FROM KERMA (NUBIA).

By MARGOT COLLETT.
(Crewdson Benington Student.)

(1) *The Discovery of the Skulls at Kerma.* The skull series forming the subject of the present paper came from Kerma, a place lying on the east bank of the Nile between Argo and Tombos (see Fig. I). Kerma is 150 miles south of the boundary which divides the Anglo-Egyptian Sudan from Egypt and as the crow flies it is almost as far from Kerma to Thebes, a distance of nearly 450 miles, as it is from Thebes to Alexandria. The excavations were carried out by Dr George A. Reisner leading the joint expedition of Harvard University and the Boston Museum of Fine Arts in the seasons 1913–14 and 1915–16, and the following particulars are taken from his report*. No certain date for the occupation of Kerma is given earlier than the Twelfth Dynasty, but objects of earlier date have been found there and the settlement is believed to be one of great antiquity. The ruins of mud houses discovered under the foundations of buildings of Middle Dynastic date show clearly that there had been a settlement of considerable size in earlier times. Taking into consideration the local history of other parts of Nubia, it is most probable that during the Old Kingdom (1st—8th Dynasties) and the Early Middle Kingdom (9th—11th Dynasties) Kerma was one of the numerous Egyptian trading stations held by an agent and a few men, possibly local natives, at which the periodic expeditions from Egypt called to deliver and collect goods. These expeditions were sent out every two or three years carrying ointment, honey, faience and cloth, which were taken in exchange for resin, ivory, woods, oils, special grain, incense and leopard skins. The discovery of statuettes and pottery on which were inscribed the names of Pepy I and II, who were kings of the Sixth Dynasty, and of Amenemhat I of the Twelfth Dynasty, gave the first real evidence for purposes of dating. It is suggested that the settlement was slightly increased in the reign of Amenemhat I after the quelling of a native revolt. Dr Reisner gives 1970 B.C. as the date of this revolt. Later, in the reign of Sesostris I (also of the Twelfth Dynasty), and probably just after another native rising, a larger force was sent out from Egypt under the governorship of Hepzefa who was probably responsible for the erection of a fortress or fortified residence known as the Western Deffûfa. This building was the nucleus of a military settlement. It was of considerable size, containing a guard-room and several other rooms, some of which may have been used as storehouses. It is thought that the Deffûfa was built principally as a stronghold for the protection of goods brought from Upper Egypt for purposes of exchange,

* "Excavations at Kerma," Vol. I (Parts I—III) and Vol. II (Parts IV—V). *Harvard African Studies*, Vols. V and VI (1923).

and of the taxes collected from the local tribes. Owing to the lack of water nearby there would not have been a convenient harbourage for a large armed force in the building.



This governor (Hepzefa), besides being the builder of the [illegible]ffûfa, is believed to have been the founder of the Egyptian Cemetery in which we are chiefly interested. It remained in use for over 350 years. In the neighbourhood of the

Deffûfa were remains of only two other buildings, both being funerary chapels, one of which was attached to the earliest tumulus (K III) and the other to one of later date (K X). The pottery and grave furniture throughout this cemetery, the painting in the second chapel, the seals, and the numerous statuettes were all distinctly Egyptian in form and technique. On the other hand there were certain peculiarities quite unknown in the contemporary graves in Egypt proper. The principal among these peculiarities was the sacrificial, or so-called sâti, burial when in some cases as many as 320 people appear to have been buried alive with the body of their chief. This custom was prevalent in Egypt during Predynastic and First Dynastic times; it was also common in the earliest Nubian graves and it appears to have been the custom at Ur in early times. It was not only practised by the ruling classes at Kerma; the smaller independent graves of quite poor type also have one or possibly two human sacrifices. In the later graves, where the Nubian element became more marked, a ram or several rams seem to have been substituted for the human sacrifices in some cases, or there might be three rams and two human beings as opposed to one ram and four human beings in an earlier grave. The covering of the body with a cowhide shows Egyptian influence, but again the custom of bed burials was unknown in Upper Egypt at this time. In one of the later tumuli coffin burials were introduced contemporaneously with bed burials, another sign of Egyptian influence. The pottery became noticeably coarser in the later graves and, although developing along the same lines as in Egypt proper, a certain degeneration is evident.

The cemeteries at Kerma can be divided into three groups: the earliest Egyptian Cemetery founded by Hepzefa, the Nubian Cemetery which follows a transition period of Egyptian-Nubian graves and the much more recent Third and Fourth Century A.D. Meroitic Cemetery. With the exception of a few skulls from the Meroitic Cemetery which have not been measured all our material comes from the earliest Egyptian Cemetery which was in use during the Twelfth and Thirteenth Dynasties. The following table gives the numbers of skulls of which measurements were taken belonging to different graves or tumuli, including the Meroitic skulls:

No. of Tumulus or Grave	No. of Skulls		Dynasty	Nature of Interment
	Main	Subsidiary		
III	11	22	12th	Tumulus
IV	13	42	"	"
X	48	56	13th	"
XVI	4	16	"	"
XVIII	11	6	"	"
XXIX	1	—	"	Minor tumulus
XX	2	—	"	" "
XXIII	10	—	"	Tumulus
XXXIV	1	—	"	Minor tumulus
XXXV	1	—	"	" "
B	9	—	12th	Graves in open
Meroitic	4	—	3rd and 4th cent. A.D.	" "

The tumuli throughout the cemeteries were all constructed on the same plan. They consisted of one or two main chambers centrally placed, one of which contained the chief body, a sacrificial corridor dividing the tumulus in half, and built at right angles to this corridor a series of parallel walls gradually decreasing in height towards the outside. The subsidiary graves were placed between these walls and they are thought to have held the bodies of the officials belonging to the court of the chief. The height of the walls where they met the walls of the corridor was about 3 ft. and it decreased to about 1 ft. on the outside edge. The whole erection was covered with a mound of earth and surrounded by a ring of ox skulls which were possibly the remains of a funeral feast, and also by a ring of black stones for which we have no explanation. In addition to these tumuli, which differed in size considerably, there were also several independent graves consisting of oval or rectangular pits and containing one, two or three bodies each. Both primary and sacrificial burials were made in each form of funerary chamber or grave and it is not possible from the records available to distinguish the two varieties in all cases. It is probable, however, that considerably more than half of the skulls dealt with were those of sacrificial victims. The tumuli were numbered in chronological order as far as this was possible. The actual skeletons from the central corridor, representing the chief body and the sacrificial bodies, were lettered, and the skeletons from the subsidiary graves numbered. Of the total 310 skulls available 55 were excluded from the series measured for various reasons and this accounts for breaks in the serial numbering which was done in the Biometric Laboratory. The present paper deals only with the skulls, though many of them have mandibles and considerable numbers of the other bones of the skeleton have been preserved.

The archaeological evidence was apparently only capable of giving a very incomplete history of the settlement at Kerma, and the account of the excavations throws little or no light on many important points. We may conclude, however, that the original trading colony, which was founded at an early but unknown date, almost certainly consisted of pure Egyptians, although there was an early non-Egyptian village of considerable size and also a fairly large Nubian village in the vicinity. The trading settlement Inebuw Amenemhat (Kerma) was known to be in existence in the Twelfth Dynasty and evidence of a settlement much earlier than this is given. Reisner says that "the colony at Inebuw Amenemhat was... on the site of an old trading post of the 6th Dynasty." After the advent of Hepzefa the number of colonists was not substantially, if at all, increased by the traders from Egypt as far as we know, although there is no definite evidence on this point. Supposing there was no increase by immigration, the occupants must have interbred and remained in possession after the Twelfth Dynasty. The tenure was a military one and it is unlikely that there was much intermarriage with native peoples of the locality. The culture, as we have seen, remained essentially Egyptian in type until a much later period than that with which we are concerned, and the fact that a few peculiarities were observed such as the common practice of sacrifical burial and the use of a bed in burial cannot be accepted as evidence of blood

admixture with any foreign element. It would appear that all these peculiarities were merely survivals of elements of an earlier Egyptian culture. The bulk of our material belongs to a comparatively short period, probably not exceeding 250 years. Reisner says: "For the period from the death of Hepzefa (K III) to that of the person buried in K XX I estimate an interval of about 200 years."

It is fortunate that our knowledge of the craniology of Upper Egypt from Early Predynastic to Roman times is quite extensive, since the original colonists almost certainly came from there. It is known that the physical type in the north was undergoing slight but continual modification, and we may hope to give a definite answer to the questions whether the community which was apparently isolated in Nubia was, during the period for which we have acquaintance with it, of the same physical type as the contemporary population of Upper Egypt, or whether it was of the same type as the Upper Egyptian population of another period, or whether it had acquired distinctive characters which would differentiate it from all known populations of Egypt.

(2) *The Nature of the Kerma Series.* There are 310 skulls from Kerma preserved in the Biometric Laboratory. Fifty-five of these were excluded from the series, and no measurements of them were taken, for various reasons: 31 are posthumously distorted, eight are too fragmentary to be measured, one is scaphocephalic, seven are immature, four are of Meroitic date, two found in tumuli are so well preserved that they can only be supposed to represent modern intrusive burials and two others possess manifest negro characters. The remaining skulls, which are the only ones considered in later sections of this paper, are all in a similar state of preservation, which resembles that of most other Predynastic and Dynastic Egyptian series, and most of them are complete or nearly complete. With two exceptions, they all bear grave numbers, which assign them to the Twelfth and Thirteenth Dynasties, and these are given with the corresponding serial numbers in the appended tables of individual measurements. The specimens which were not marked by the excavators (Nos. 224 and 272) are presumed to belong to the same period as the others. When the skulls were set out on a table so that all could be seen together, few striking differences in type were observed except in the case of the two with clear negro characters which were excluded from the series on this account. It was evident that if any division of the remainder was to be made by merely examining the skulls, such would have to depend on the fact that some possess more negroid characters—viz. a flatter nasal bridge, a higher nasal index and a greater degree of prognathism—than others. But an attempt to divide the whole into two contrasted groups showed that it was quite impossible to distinguish the negroid and non-negroid specimens with any degree of exactness. Hence it was concluded that the safest procedure was to treat the total series as if it represented a single racial type which would obviously be one possessing negroid characters. The series did not appear to be more heterogeneous than many which have been dealt with in this way.

The sexing of the 255 adult skulls forming the series was done by Professor

Karl Pearson and Dr G. M. Morant. They distinguished 141 males and 114 females. This slight excess of the representatives of the former sex over those of the latter has frequently been found for a collection of excavated crania, and it has been supposed due to the fact that the weaker female specimens would be broken more easily and hence less likely to be saved by the excavators. In the present case many of the skulls are known to be those of sacrificial victims and this may also account in part for the presumed disparity between the sexes. Sex ratios, i.e. male means divided by female means, have been given by Kitson* for two European, two negro and the long Egyptian *E* series of 26th—30th Dynasty skulls. The Kerma values are closest to those for the Egyptians and Bantu negroes (Teita) from Kenya Colony and comparisons with these are made in our Table I. The correspondence of the sex ratios for these three series is extremely close and we may conclude that they were all sexed in a uniform way.

TABLE I.

Sex Ratios for the Kerma and other Series.

Character	Kerma	Egyptian *E*	Teita
L	1·045	1·047	1·046
B	1·029	1·025	1·024
LB	1·054	1·053	1·051
B′	1·040	1·027	1·039
H	1·058	1·038	1·044
S	1·039	1·034	1·037
U	1·044	1·038	1·037

A rough estimate of the age constitution of the cemetery population represented can be obtained by considering the state of closure of the coronal, sagittal and lambdoid sutures. The results for the Kerma series are summarised below, those three calvarial sutures being the only ones referred to.

Sex	All sutures open	Sutures beginning to close or partly closed	All sutures closed	Totals
♂	16 (11·3 %)	83 (58·9 %)	42 (29·8 %)	141
♀	40 (35·1 %)	51 (44·7 %)	23 (20·2 %)	114

The percentages may be compared with those given† for six other series examined by Dr Morant. For every one of these marked differences were found between the corresponding male and female frequencies, confirming the fact that the sutures close at a later age for women than for men. The Kerma male percentage of skulls showing all three sutures open is lower than any previously given, and the percentage with all sutures closed is only exceeded by one other. The females show a smaller proportion with all sutures open and a larger with all sutures closed than has been found for any other series examined. The present

* *Biometrika*, Vol. XXIII (1931), p. 275.

† *Biometrika*, Vol. XXIV (1932), p. 170.

sample must hence be supposed to represent an older cemetery population than any of the others, although the contrary would have been expected owing to the fact that many of the people buried at Kerma met unnatural deaths as they were sacrificial victims.

We may enquire next whether there is any evidence of a change in the racial constitution of the population of Kerma during the Twelfth and Thirteenth Dynasties. The male means were computed for each of these Dynasties and the crude coefficient of racial likeness of $0{\cdot}92 \pm {\cdot}17$ for 31 characters was found between them. No significant differences between the means were found. The coefficient differs significantly from zero, but it may be too high owing to the fact that the Egyptian *E* standard deviations were used in computing it in place of the standard deviations for each of the samples compared. In any case it is of such a low order that it was thought safest to conclude that there is no evidence of a change in the racial nature of the Egyptian settlers at Kerma during the period considered. The female coefficient was not calculated, but a few of the more important means for the two Dynasties are compared in the table below.

Character	Male		Female	
	Twelfth Dynasty	Thirteenth Dynasty	Twelfth Dynasty	Thirteenth Dynasty
100 *B*/*L*	71·8 (59)	72·5 (74)	73·1 (37)	73·4 (73)
100 *B*/*H'*	101·0 (41)	100·2 (50)	101·8 (27)	101·9 (59)
100 *NB*/*NH*, *R*	51·5 (48)	51·6 (63)	53·0 (28)	53·4 (55)
N ∠	64°·3 (45)	65°·4 (50)	67°·8 (21)	67°·4 (42)

The differences between these means in the case of each sex are quite insignificant. All the Twelfth and Thirteenth Dynasty skulls from Kerma accordingly were pooled, and the means, standard deviations and coefficients of variation with their probable errors for the total sample are given in Table II for all the characters measured and for the indices and angles*. I was able to make use of a considerable

* [Definitions of the measurements, which are denoted by the usual index letters, will be found in *Biometrika*, Vol. xx^{B} (1928), pp. 362—364. A single change was made in the way the nasal height *NH'* was measured. In recent craniometric studies published in *Biometrika* this has been taken, in addition to the Frankfurt nasal heights, from the nasion to the "base" of the anterior nasal spine with the object of providing a measurement which was supposed the same as the one defined by Broca and Martin and again in the Monaco scheme. While the definitions provided in these techniques are not perfectly clear, a re-examination suggested that a closer approach to the nasal height actually taken by those who have followed them could be obtained in the following way. The lower margins of the pyriform aperture are first drawn as pencil lines and the Frankfurt heights are taken to these curved lines on either side. A third pencil line, which will be horizontal on a symmetrical specimen, is then drawn so that it appears, when the skull is viewed in *norma facialis*, to be a straight line joining the lowest points of the margins on either side. The point where this horizontal line meets the boundary of the anterior nasal spine—itself a vaguely defined line—on the left-hand side is supposed to be Martin's nasospinale. The *NH'* given for the Kerma skulls is the chord from this point to the nasion, and in all cases it was found to be very close to the Frankfurt *NH*, *L*. The *NH''*s given by Miss Kitson for the Naga skulls in the present volume of *Biometrika* were taken in the same way, but in earlier studies in the Biometric Laboratory the *NH'* was to the "base" of the anterior nasal spine.—G. M. M.]

number of measurements which had previously been taken by Mr G. C. Dunning and the individual measurements are given in Tables VIII, IX. The nature of the sample can be estimated from the constants in Table II. It may be asked first

TABLE II.

Constants of the Male and Female Kerma Series.

Character	Means		Standard Deviations		Coefficients of Variation	
	Male	Female	Male	Female	Male	Female
C	1383·3 ± 8·2 (72)	1229·9 ± 7·5 (47)	103·5 ± 5·8	86·8 ± 5·3	7·48 ± ·39	7·06 ± ·42
F	183·3 ± ·38 (138)	177·0 ± ·36 (114)	6·63 ± ·27	5·79 ± ·27	3·62 ± ·15	3·27 ± ·15
L	185·2 ± ·36 (138)	177·2 ± ·35 (114)	6·26 ± ·25	5·68 ± ·25	3·38 ± ·14	3·20 ± ·14
B	133·6 ± ·26 (136)	129·8 ± ·28 (112)	4·52 ± ·18	4·37 ± ·20	3·38 ± ·14	3·37 ± ·15
B′	92·8 ± ·23 (139)	89·2 ± ·27 (111)	4·08 ± ·16	4·20 ± ·19	4·40 ± ·19	4·71 ± ·21
B″	113·1 ± ·34 (68)	107·5 ± ·42 (60)	4·13 ± ·24	4·85 ± ·30	3·65 ± ·21	4·51 ± ·28
Biasterionic *B*	104·9 ± ·29 (113)	100·2 ± ·24 (102)	4·51 ± ·20	3·60 ± ·17	4·26 ± ·19	3·59 ± ·17
H′	133·4 ± ·32 (93)	127·8 ± ·31 (88)	4·65 ± ·28	4·38 ± ·22	3·48 ± ·17	3·43 ± ·17
H	135·7 ± ·34 (100)	128·2 ± ·35 (77)	5·01 ± ·24	4·56 ± ·25	3·69 ± ·18	3·56 ± ·19
OH	110·7 ± ·26 (125)	107·4 ± ·25 (91)	4·25 ± ·18	3·48 ± ·17	3·84 ± ·16	3·24 ± ·16
LB	101·8 ± ·26 (117)	96·6 ± ·27 (89)	4·16 ± ·18	3·75 ± ·19	4·09 ± ·22	3·88 ± ·19
S_1'	112·1 ± ·30 (126)	107·4 ± ·29 (109)	4·96 ± ·21	4·53 ± ·21	4·42 ± ·19	4·22 ± ·19
S_2'	115·1 ± ·34 (128)	111·0 ± ·35 (106)	5·78 ± ·24	5·26 ± ·24	5·02 ± ·22	4·74 ± ·22
S_3'	97·2 ± ·32 (124)	94·3 ± ·31 (97)	5·34 ± ·23	4·58 ± ·22	5·49 ± ·24	4·86 ± ·23
S_1	128·1 ± ·38 (126)	123·3 ± ·39 (109)	6·28 ± ·27	6·05 ± ·28	4·90 ± ·21	4·91 ± ·22
S_2	128·7 ± ·44 (128)	124·0 ± ·46 (106)	7·46 ± ·31	7·06 ± ·33	5·80 ± ·25	5·69 ± ·23
S_3	116·8 ± ·46 (124)	112·9 ± ·46 (97)	7·67 ± ·33	6·71 ± ·32	6·57 ± ·28	5·94 ± ·29
S	374·7 ± ·77 (133)	360·3 ± ·80 (105)	13·15 ± ·54	12·13 ± ·56	3·51 ± ·15	3·37 ± ·16
U	510·2 ± ·81 (133)	488·9 ± ·83 (107)	13·89 ± ·57	12·78 ± ·54	2·72 ± ·09	2·61 ± ·12
Q′	305·0 ± ·54 (123)	291·8 ± ·60 (89)	8·98 ± ·37	8·33 ± ·42	2·94 ± ·13	2·85 ± ·14
fml	36·4 ± ·20 (112)	34·6 ± ·15 (92)	3·13 ± ·14	2·14 ± ·11	8·60 ± ·39	6·18 ± ·30
fmb	30·1 ± ·16 (110)	28·4 ± ·16 (83)	2·42 ± ·11	2·12 ± ·11	8·04 ± ·37	7·46 ± ·39
G′H	69·6 ± ·29 (111)	65·7 ± ·25 (83)	4·48 ± ·20	3·33 ± ·17	6·45 ± ·29	5·07 ± ·27
GL	96·5 ± ·32 (97)	93·7 ± ·38 (68)	4·75 ± ·23	4·63 ± ·27	4·92 ± ·24	4·94 ± ·29
GB	95·3 ± ·28 (111)	90·9 ± ·30 (87)	4·64 ± ·21	4·20 ± ·21	4·87 ± ·22	4·62 ± ·24
J	127·5 ± ·38 (83)	118·9 ± ·39 (52)	5·16 ± ·27	4·16 ± ·28	4·05 ± ·24	3·50 ± ·23
NH, R	50·0 ± ·18 (112)	47·2 ± ·20 (86)	2·82 ± ·14	2·79 ± ·14	5·64 ± ·25	5·91 ± ·30
NH, L	50·1 ± ·19 (112)	47·3 ± ·20 (86)	3·05 ± ·14	2·70 ± ·14	6·09 ± ·28	5·71 ± ·30
NH′	50·0 ± ·19 (105)	47·1 ± ·19 (90)	2·95 ± ·15	2·73 ± ·14	5·90 ± ·27	5·80 ± ·29
NB	25·8 ± ·11 (114)	25·0 ± ·15 (89)	1·79 ± ·08	2·13 ± ·11	6·94 ± ·31	8·52 ± ·43
DS	11·6 (32)	11·1 (22)	—	—	—	—
DC	22·7 (33)	22·2 (23)	—	—	—	—
DA	34·5 (32)	33·0 (22)	—	—	—	—
SS	4·0 ± ·09 (95)	3·1 ± ·09 (69)	1·38 ± ·07	1·12 ± ·06	34·50 ± 1·88	36·13 ± 2·33
SC	10·8 ± ·13 (95)	10·2 ± ·17 (72)	1·99 ± ·10	2·15 ± ·12	18·43 ± ·93	21·08 ± 1·28
G_1	52·0 ± ·28 (87)	49·7 ± ·26 (59)	3·82 ± ·14	3·01 ± ·19	7·35 ± ·38	6·06 ± ·38
G_1'	47·3 ± ·21 (88)	45·7 ± ·24 (62)	2·94 ± ·15	2·84 ± ·17	6·22 ± ·32	6·21 ± ·38
G_2	39·4 ± ·22 (82)	37·8 ± ·23 (62)	2·90 ± ·15	2·71 ± ·16	7·36 ± ·39	7·17 ± ·44
EH	13·2 ± ·18 (55)	12·3 ± ·18 (47)	2·03 ± ·13	1·85 ± ·13	15·38 ± 1·01	15·04 ± 1·07
O_1, *R*	41·6 ± ·11 (116)	39·7 ± ·12 (86)	1·77 ± ·08	1·60 ± ·08	4·25 ± ·19	4·03 ± ·21
O_1, *L*	41·5 ± ·11 (105)	39·7 ± ·11 (84)	1·73 ± ·08	1·44 ± ·07	4·17 ± ·19	3·63 ± ·19
O_1', *R*	39·4 ± ·19 (45)	37·7 ± ·14 (47)	1·91 ± ·13	1·47 ± ·10	4·85 ± ·35	3·90 ± ·27
Lacrymal O_1, *R*	38·2 (28)	36·6 (40)	—	—	—	—
O_2, *R*	32·7 ± ·13 (114)	32·5 ± ·13 (83)	2·06 ± ·08	1·89 ± ·09	6·30 ± ·28	5·81 ± ·30
O_2, *L*	32·7 ± ·14 (103)	32·5 ± ·14 (85)	2·06 ± ·10	1·90 ± ·09	6·30 ± ·30	5·84 ± ·30

TABLE II (*continued*).

Character	Means		Standard Deviations	
	Male	Female	Male	Female
100 B/L	72·2 ± ·20 (133)	73·3 ± ·19 (112)	3·38 ± ·14	3·01 ± ·14
100 H'/L	72·4 ± ·19 (91)	72·4 ± ·17 (88)	2·67 ± ·13	2·37 ± ·12
100 H/L	73·2 ± ·18 (100)	72·9 ± ·18 (77)	2·61 ± ·12	2·33 ± ·13
100 B/H'	100·6 ± ·31 (91)	101·9 ± ·32 (88)	4·34 ± ·22	4·50 ± ·23
100 B/H	99·0 ± ·30 (98)	101·6 ± ·34 (77)	4·42 ± ·21	4·44 ± ·24
100 $(B—H')/L$	0·5 ± ·23 (89)	1·3 ± ·24 (88)	3·05 ± ·15	3·29 ± ·17
100 $G'H/GB$	72·7 ± ·33 (104)	72·2 ± ·33 (79)	5·00 ± ·23	4·32 ± ·23
100 NB/NH, R	51·6 ± ·25 (111)	53·2 ± ·35 (84)	3·94 ± ·18	4·72 ± ·25
100 NB/NH, L	51·6 ± ·25 (111)	53·3 ± ·34 (85)	3·95 ± ·18	4·78 ± ·24
100 NB/NH'	51·8 ± ·27 (105)	53·4 ± ·36 (85)	4·16 ± ·19	4·87 ± ·25
100 SS/SC	37·3 ± ·78 (95)	30·5 ± ·72 (69)	11·22 ± ·55	8·85 ± ·51
100 DS/DC	51·3 (32)	50·7 (22)	—	—
100 G_2/G_1	75·7 ± ·53 (67)	75·5 ± ·65 (42)	6·39 ± ·37	6·26 ± ·46
100 G_2/G_1'	83·6 ± ·58 (67)	82·5 ± ·71 (44)	7·04 ± ·41	7·00 ± ·50
100 EH/G_2	33·3 ± ·53 (55)	32·8 ± ·52 (47)	5·81 ± ·37	5·29 ± ·37
100 O_2/O_1, R	78·6 ± ·32 (113)	81·6 ± ·33 (83)	5·06 ± ·23	4·49 ± ·24
100 O_2/O_1, L	78·9 ± ·32 (104)	81·8 ± ·33 (84)	4·85 ± ·23	4·44 ± ·23
100 O_2/O_1', R	81·5 ± ·52 (44)	84·9 ± ·47 (46)	5·14 ± ·37	4·74 ± ·33
100 O_2/Lacr. O_1, R	84·0 (28)	87·7 ± ·52 (40)	—	4·84 ± ·36
100 fmb/fml	83·2 ± ·41 (106)	82·0 ± ·39 (81)	6·23 ± ·29	5·92 ± ·31
Oc. I.	60·4 ± ·15 (124)	60·6 ± ·16 (97)	2·50 ± ·11	2·37 ± ·11
$N\angle$	64°·9 ± ·21 (95)	67°·6 ± ·30 (63)	3°·08 ± ·15	3°·52 ± ·21
$A\angle$	74°·1 ± ·24 (95)	72°·1 ± ·27 (63)	3°·52 ± ·17	3°·22 ± ·19
$B\angle$	41°·0 ± ·20 (95)	40°·3 ± ·18 (63)	2°·85 ± ·14	2°·15 ± ·13
Alveolar $P\angle$	83°·8 ± ·27 (69)	83°·6 ± ·45 (54)	3°·38 ± ·19	4°·96 ± ·32
Prosthion $P\angle$	81°·8 ± ·27 (88)	81°·8 ± ·46 (59)	3°·73 ± ·19	5°·21 ± ·32
$\theta_1\angle$	31°·0 ± ·22 (62)	28°·1 ± ·52 (45)	2°·62 ± ·16	5°·16 ± ·37
$\theta_2\angle$	10°·4 ± ·27 (62)	12°·3 ± ·53 (45)	3°·11 ± ·19	5°·25 ± ·37

whether the male and female series represent the same racial type. The differences of the corresponding mean indices and angles for the two sexes are found to exceed three times their probable errors in the case of 100 B/L (Δ/p.e. $\Delta = 4{\cdot}0$), 100 SS/SC (6·4), 100 O_2/O_1, R (6·5), 100 O_2/O_1, L (6·3), 100 O_2/O_1', R (4·8), 100 B/H (5·7), 100 NB/NH, R (3·7), 100 NB/NH, L (4·0), 100 NB/NH' (3·6), $N\angle$ (7·4), $A\angle$ (5·5), $\theta_1\angle$ (5·1) and $\theta_2\angle$ (3·2). Sexual differences of the same sign as those now found are to be expected in the case of the first five of these indices. The difference in the case of 100 B/H may be due to chance causes as that for the very similar index 100 B/H' is insignificant. The other measurements selected in this way are the nasal indices, which all measure the same character in slightly different ways, and four angles which are necessarily related. They simply show that the female type has a significantly higher nasal index and a greater degree of prognathism (judging by the nasal but not by the profile angles) than the male*. This might suggest that there was more negro blood among the women than among the men

* These relations are still true if the Twelfth and Thirteenth Dynasties are considered separately, as is shown by the table on p. 260. But the sexual differences between the corresponding means there are probably all insignificant.

of Kerma, but it would be rash to accept such a conclusion without further evidence, and comparisons of a different kind made below (p. 272) fail entirely to substantiate it. The fact that the majority of the measurements of shape show either insignificant differences, or differences which would have been expected, must be taken to indicate that the male and female samples really represent the same racial type.

Sexual differences in variability may be considered next and comparisons were made between the coefficients of variation of the absolute measurements and the standard deviations of the indices and angles. For these constants 48 male values exceed the corresponding female values and for the remaining 19 the female are in excess. For the long Egyptian *E* series the male variabilities are the greater in 46 cases and the female in the other five, but a proportion more similar to that found for the present series has been observed for most other long cranial series available. The only significant difference found for the Kerma male and female coefficients of variation is for *fml* (Δ/p.e. $\Delta = 4{\cdot}9$) and for the standard deviations of the indices and angles the only significant differences (Δ/p.e. $\Delta > 3$) are in the case of the *Alveolar* $P\angle$ (4·2), *Prosthion* $P\angle$ (4·0), $\theta_1\angle$ (6·0) and $\theta_2\angle$ (4·7). Finally we may compare the Kerma variabilities with those given for other series and it will suffice to consider only the Egyptian *E* series in this connection. It is known to be more homogeneous than almost all others that have been measured. Still comparing the coefficients of variation of absolute measurements and the standard deviations of indices and angles, it is found that 31 of the male Kerma variabilities are greater than the Egyptian *E* and 15 less, while four of the differences exceed three times their probable errors; for the females the Kerma variabilities are the greater in 26 cases, the lesser in 19 and there is equality in one case, while 10 of the differences may be considered significant. The Kerma series is thus rather more variable than the Egyptian *E*, but it is as homogeneous as many which have to be accepted as representing single and indivisible racial types.

(3) *Remarks on the Condition and Anomalies of the Kerma Skulls.* The crania from Kerma were examined for anomalies after the method customarily employed in the Biometric Laboratory in recent years. There are totals of 141 male and 114 female specimens*, but owing to the incomplete nature of several of the skulls the totals which might have been affected are recorded separately for several of the more important anomalies.

(*a*) *Sutures.* Remarks on the condition of the coronal, sagittal and lambdoid sutures for each skull are given in the appended tables of individual measurements. Unless otherwise stated it may be assumed that any one, or all, of these sutures are open for their entire lengths. The approximate estimate of the age constitution of the sample which can be deduced from these data has been considered above. Another point of interest is the order of closure of the three

* An adult female skull (No. 254) is not included in this total and it was not measured since it appears to be scaphocephalic. The sagittal suture is completely obliterated, the coronal is closing and the lambdoid clearly open for its whole length.

principal sutures since a clear racial difference is supposed to be shown in this respect. The following table gives the frequencies found for the different orders of closure:

Sex	Sagittal closing before coronal and lambdoid	Sagittal and coronal closing together before lambdoid	Sagittal and lambdoid closing together before coronal	Coronal closing before sagittal and lambdoid	Lambdoid closing before sagittal and lambdoid	Totals	Coronal closing before lambdoid	Lambdoid closing before coronal
♂	75	20	3	18	1	117	79	20
♀	34	18	2	14	2	70	48	10

It is apparent from these figures that the sagittal shows a clear tendency to close before the other two sutures, while the coronal also tends to close before the lambdoid suture. The order of closure found with the greatest frequency in the case of both sexes is sagittal—coronal—lambdoid. Three other series have previously been examined in the Biometric Laboratory in exactly the same way and it was found that for negroes (Teita) the coronal suture showed a slight tendency to close before the sagittal in the case of the males, and a definite tendency in the case of the females, though the numbers (37 males and 19 females) on which these estimates were based are small. Both the coronal and sagittal sutures showed a marked tendency to close before the lambdoid. In the Hythe and Spitalfields English series the sagittal suture was found almost invariably to be the first to close, or to begin closing, followed by the coronal and lambdoid sutures which began closing at approximately the same time. Thus the Kerma type occupies an intermediate position between the Teita—a negro series—on the one hand and the Hythe and Spitalfields European series on the other. For the negroes the coronal suture closes before the sagittal and well before the lambdoid; for the Nubians the coronal suture closes after the sagittal but before the lambdoid again, and for the Europeans the coronal suture closes after the sagittal but almost at the same time as the lambdoid. These are only average results, and the numbers on which they are based are unfortunately small, but in confirmation of Gratiolet's Law they appear to indicate a definite racial difference in the order of sutural closing. The figures for the series examined in the Biometric Laboratory suggest that the tendency for the coronal suture to close before the lambdoid is rather more marked in the case of females than in the case of males for any particular racial series. As the numbers for the Kerma series are fairly adequate, it was considered worth while to examine whether the order of closure of the coronal and lambdoid sutures is associated with the magnitude of breadth measurements of the calvaria. The following table giving measurements for the two groups of male skulls from Kerma shows that the differences between the corresponding means are clearly insignificant:

Order of suture-closing	*B'*	*B*	Biasterionic *B*	*L*
Coronal suture closing before lambdoid	90·8 (72)	133·6 (67)	105·4 (57)	185·7 (67)
Lambdoid suture closing before coronal	93·8 (18)	134·8 (19)	105·7 (13)	185·7 (18)

Complete cases of metopism were found in four male out of a possible 89 skulls, and six female out of a possible 97 skulls, only small traces of a suture above the nasion being found in other specimens. All the longer European series to be examined have shown a percentage of about 10 for metopic sutures and in the case of negro series the frequency is about 1 per cent., among the 122 Teita adult skulls only one metopic suture being found. The Kerma skulls, therefore, with a percentage of 4·5 for the males and 6·2 for the females, again occupy an intermediate position between the negro and white races. Contact between the frontal and parietal bones in the case of a metopic suture being present is indicated by $LF+RP$ or $RF+LP$ and the measurement given is the length of the common suture. The males affected are Nos. 107 (contact?), 111 (at bregma), 113 (at bregma), and 126 ($?\,RF+LP$); and the females Nos. 163 ($LF+RP$, 3·1 mm.), 165 ($RF+LP$, 3·2 mm.), 166 ($LF+RP$, 8·2 mm.), 202 ($LF+RP$, 5·7 mm.), 219 ($LF+RP$, 2·9 mm.) and 224 ($LF+RP$, 7·7 mm.). In accordance with what has been previously found, contact between the left frontal and right parietal bones is the most frequent occurrence. It has been stated that when the metopic suture persists to an adult stage it closes at about the same age as the sagittal. Five out of the ten metopic Kerma skulls have both frontal and sagittal sutures open; in two other cases both are closed and in the remaining three the sagittal suture is open but the frontal is closed. One female specimen (No. 202) has the sagittal suture open but the frontal entirely closed and partly obliterated. Only when the squamous process made actual contact with the frontal bone were the cases of fronto-temporal articulation noted, these being in the males Nos. 62, 69, 136, 138 and 147 (on both sides), 70 and 124 (on the left but not on the right side), 3 and 51 (on the left while the right side was too defective to permit examination); and in the females Nos. 152, 194, 195, 199, 255 and 271 (on both sides), and 160, 173, 203 and 230 (on the right but not on the left side). For the males there were 89 cases on the right side where the sutures at the pterion were sufficiently open to permit examination for cases of fronto-temporal articulation and of these five were affected giving a percentage of 5·6, and on the left nine were affected out of a possible 101 (8·9 °/₀); for the females, on the right side 10 were affected out of a total of 83 (12·0 °/₀), and on the left six out of a possible 92 (6·5 °/₀). These percentages are unusually high, as for European series in general the frequency is of the order of 1·6 per cent., but for negroes an average of 12 per cent. has been found*. The Kerma series it will be noticed once again occupies an intermediate position. No complete case of a horizontal suture across the malar bone occurs in the series, two female skulls however (Nos. 197 and 218) show slight traces of it. No traces of sutures between the ex- and supra-occipitals were observed.

(*b*) *Supernumerary Bones.* In the male series four cases of true interparietal bones were found out of a possible 111 skulls (3·6 °/₀) and for the females only one out of a possible 104 cases (0·9 °/₀) was noted, thus giving a percentage of 2·3 for

* Cf. le Double, *Traité des Variations des Os du Crâne de l'Homme* (1903), pp. 302—303.

the whole series. Owing to the irregularity of the percentages found for both European and negro series in this respect, it is not known whether any racial significance can be attached to the Kerma figures. Of the male specimens, No. 36 shows the *os pentagonale* and right *os triangulare* separate but with no suture between them; the other three skulls (Nos. 98, 121 and 130) have the *os pentagonale* only separate. The female specimen (No. 198) also has the *os pentagonale* only separate. An *os épactal** was noted in one male skull (No. 41): it is divided by a vertical suture which is to the right of the sagittal suture. There are no examples of an ossicle of bregma and only small Wormian bones were found in the coronal and sagittal sutures. The Wormian bones in the lambdoid suture appeared to be smaller and less frequent than in European series. Two unusual anomalies were noted; a male specimen (No. 107) has a large Wormian bone between the temporal squama and parietal on the left side and a female specimen (No. 153) has two large and two small Wormian bones in the same position on the left while the right side is normal (see Plate IV *b*). Several cases of an ossicle of lambda were observed. Records were made of the epipteric bones found, only those with a maximum diameter greater than 3 mm. being counted. In compiling the total numbers which might have possessed these supernumerary bones all those skulls were excluded on which the superior border of the greater wing of the sphenoid was indefinite on the side considered owing to synostosis. More males than females were excluded on this account. The following frequencies were found for the group having the region of the pterion intact on both sides and with the sutures there visible:

Sex	No epipteric bones	Epipteric bones (one or more) on both sides	One or more epipteric bones on the right but none on the left	One or more epipteric bones on the left but none on the right	Totals
♂	40	4	4	5	53
♀	43	9	7	9	68

A considerable difference is shown between the percentages for the two sexes, the males having 13 skulls with one or more epipteric bones out of a possible 53 cases (24·5 %), and among the females there are 25 cases out of a possible 68 specimens (36·8 %). According to le Double† it is not known for which sex the anomaly occurs with the greater frequency and comparison with the Hythe and Teita series does not encourage the assumption that there are any sexual or racial differences. The percentage for the Teita male series is 22·2 (or 8 out of a possible 36 cases) and for the female 19·5 (or 8 out of a possible 41 cases). For the Hythe skulls the male percentage is 32·0 and the female 25·8.

(*c*) *Teeth.* The state of preservation in which the teeth of the Kerma skulls were found was remarkably good, and there was a very high percentage of palates

* Cf. le Double, *Traité des Variations des Os du Crâne de l'Homme* (1903), p. 60.

† *Op. cit.* p. 306.

with no teeth lost before death. There are 111 complete male palates of which 71 have no teeth lost before death (64·0 °/₀), and for the females the percentage is 69·5, 57 skulls from a total of 82 complete palates not having lost any teeth before death. Excluding a few cases, where it could not be determined whether the third molars had erupted or not owing to the loss of one or more molars before death, an investigation with regard to the presence or absence of third molars was made with the following results:

Sex	Region of third molars complete and undeformed on both sides	No third molar on either side	Third molar on right but not on left side	Third molar on left but not on right side	Percentage having one or both third molars missing
♂	96	9	3	1	13·5
♀	73	2	0	5	9·6

Considering the sides separately, in which case a few additional specimens with the region of the third molar complete on one side but defective on the other can be included, it is found for the males that on the right side there is an absence of the third molar in 12 cases out of a possible 102 (11·8 °/₀) and on the left there are 12 showing an absence out of a possible 99 (12·1 °/₀). For the females there is absence in seven cases out of a possible 79 (8·9 °/₀) on the right and four out of a possible 82 (4·9 °/₀) on the left. Several dental anomalies were observed among the male skulls: No. 79 has only a single premolar on the right, No. 92 (Plate VI *a*) apparently lacked two teeth other than molars, No. 123 has no canine on the right, while No. 128 lacks a canine on the left side. The second right incisor in skull No. 101 has erupted behind its normal position. A curious condition is present in No. 107 (Plate VI *d*) which apparently had only three incisors of which one was placed centrally so that it appears to have erupted between the premaxillary bones. The socket for the third incisor present on the left in skull No. 4 is the only case of an extra tooth among the male specimens. The female specimen No. 155 (Plate VI *c*) has no canine present on the left side and no third molar on the right. An absence of lateral incisors was noted in skull No. 216 (Plate VI *b*), and No. 266 has only one incisor on the left and no canine or third molar on the right. Several examples were observed of crowding of the teeth and of deformations of the alveolar margin due to disease (see Plate VI *a* showing an opening in the region of the first and second left molars).

(*d*) *Other Anomalies.* The relative sizes of the jugular foramina were compared with the following results:

Sex	*JR*	*J=*	*JL*
♂	74 (65 °/₀)	19 (17 °/₀)	20 (17 °/₀)
♀	48 (62 °/₀)	9 (12 °/₀)	20 (26 °/₀)

These results are quite in accordance with those for all series previously examined in this way. Precondyles, both single median and double, were noted in several cases, a female skull (No. 194) with a single median precondyle having the largest. There are no marked cases of a *fossa pharyngea*. A female specimen (No. 180) has a large para-mastoid process on the left side with an articular surface, and one case of a basi-occipital incisure was observed on the left side of another female skull (No. 193). Plate V *b* illustrates a curious condition present in a male skull (No. 36); this is a constriction of the inferior part of the basi-occipital where it unites with the sphenoid. The numbers of tympanic perforations were recorded but no details are given as no racial or sexual differences, or evidence that they were more numerous on one side than on the other, could be deduced. What appears to be an infantile condition of some interest is present in a female skull (No. 216: Plate V *d*) which shows a complete failure to unite on the superior and inferior margins of the tympanic plate on the left side and a union only just made on the right*. None but very small exostoses were found. Three healed wounds were observed among the female skulls from a total of 114, but, as is to be expected, the wounds on the male skulls were more numerous, totalling 16 for the 141 skulls. Those of greatest interest are on male specimens and photographs of two of these are given: No. 74 (Plate IV *a*) has a depressed wound on the right parietal and temporal squama and a healed fracture of the left malar bone; No. 64 (Plate V *c*) has what appears to be a wound on the right maxillary bone below the orbit. No. 7 shows a fracture and subsequent rejoining of the left zygomatic arch. One case of a wound on the nasal bones is noted in skull No. 66. Traces of roughening of the glenoid fossa due to arthritis were found on several skulls of which the males Nos. 46, 119 and 146, and the females Nos. 151 and 236 were the most marked. The only other sign of disease was observed on the female skull No. 274 (Plate V *a*); the diseased parts are two symmetrically placed areas on either side of the lambda, the lambdoid suture being prematurely closed where it lies within the diseased areas. This may be a case of periostitis. Holes similar to those found by Prof. Elliot Smith on Nubian skulls† and supposed due to insects were evident on several of the Kerma skulls (see Plate III *b* below). The holes are mostly small and not nearly so numerous as on the skull he figures. No signs of healing are shown on our specimens and the excavations were almost certainly made after death.

(4) *Comparisons between the Kerma and other Racial Series.* In order to gain a knowledge of the racial relationships of the Kerma series, the Coefficients of Racial Likeness were calculated between its means and those available for a number of other series. With the usual notation, the form of the crude coefficient used is

$$\frac{1}{m}\Sigma\frac{(M_s - M_{s'})^2}{\sigma_s^2}\times\frac{n_s n_{s'}}{n_s + n_{s'}} - 1 \pm \cdot 67449\sqrt{\frac{2}{m}} = \frac{1}{m}\Sigma(\alpha) - 1 \pm \cdot 67449\sqrt{\frac{2}{m}}.$$

* The adolescent *Sinanthropus* skull shows a similar fissure of the tympanic plate, and the condition is normal for infants to-day.

† *The Archaeological Survey of Nubia, Report for* 1907—1908, Cairo (1910), Vol. II, p. 290, and *Plates Accompanying Vol. II*, Plate XXX, Fig. 2. See also "The alleged Discovery of Syphilis in Prehistoric Egyptians," *The Lancet*, August 22 (1908), p. 521.

The reduced coefficient is defined to be

$$50 \times \frac{\bar{n}_s + \bar{n}_{s'}}{\bar{n}_s \bar{n}_{s'}} \left[\frac{1}{m} \Sigma(\alpha) - 1 \pm \cdot 67449 \sqrt{\frac{2}{m}} \right],$$

where $\bar{n}_s$ and $\bar{n}_{s'}$ are the mean numbers of skulls available for the characters used in computing the coefficient for the first and second series in the comparison, respectively. The reduced coefficient is supposed to give the best measure of the absolute divergencies between the types which it is possible to find at present. The standard deviations of the long Egyptian *E* series of Twenty-sixth to Thirtieth Dynasty skulls were used in computing the coefficients.

The male and female Kerma series are of approximately equal length, but this is not the case for the majority of cranial series for which measurements have been provided. In general the male series are longer than the corresponding female ones and there are far more of them available. Our estimates of racial affinity are thus based principally on the comparisons of the male series. There are 20 of these series of sufficient length representing different sections of the population of Egypt from Early Predynastic to Roman times. Of the sites represented Gizeh and Deshasheh with Medum are in Lower Egypt, but all the others are in Middle or Upper Egypt. Table III gives all the coefficients of racial likeness between these Egyptian and the Kerma series and the reduced male values range from 1·11 to 19·56. Working on data provided by Morant, Woo has given* the lowest reduced coefficients of racial likeness found for each of 23 male Egyptian series, and they range approximately from $-0{\cdot}5$ to $+4{\cdot}5$. Three of the values now found with the Kerma series are less than 4·5 and two are as low as any which it has generally been possible to find for these types. We are thus led to conclude that the population of Kerma in the Twelfth and Thirteenth Dynasties was of typical Egyptian type. It has been shown by Morant† that in Upper Egypt the character of the population was changing slowly from Early Predynastic to Roman times, the type having gradually lost its original negroid characters and at the same time increased its calvarial breadth and hence its cephalic index. This general tendency was illustrated by the majority of the series available, and the exceptions to it which were found could be supposed due to the peculiarities of local types or possibly to a local and restricted admixture of part of the population with alien immigrants. The question to what period the Upper Egyptian series which most closely resemble the Kerma belong is one of particular interest. Its closest connections are found with the Naqada series which is probably of Middle Predynastic data and with a Late Predynastic series measured by Thomson and MacIver‡. A First Dynasty series is next in order and then the Kerma type is

* *Biometrika*, Vol. XXII (1930), p. 75. † *Ibid.* Vol. XVII (1925), pp. 1—52.

‡ For the series measured by Thomson and MacIver only 14 of the complement of 31 characters used in computing the coefficient of racial likeness are given. For these 14 characters the male crude coefficient between the Kerma and Naqada *A* and *Q* series is 1·21 ± ·25 and the reduced value is 1·49 ± ·31. These only differ slightly from the values in Table III given for the total 31 characters, and it may be assumed that the 14 characters provided by Thomson and MacIver lead to close approximations to the coefficients which would be obtained if all 31 characters could be used in comparisons with their series. It may be concluded that the Kerma series is equally related to the Naqada and to the Late Predynastic series of Thomson and MacIver.

TABLE III.

Coefficients of Racial Likeness between the Kerma (12th and 13th Dynasties) and other Series.*

Locality or Race	Period	Measured by	$\bar{n}$		Kerma $\bar{n}$		Crude Coefficients		Reduced Coefficients	
			♂	♀	♂	♀	♂	♀	♂	♀
Naqada (*A* and *Q* series)† ...	Predynastic	Fawcett	65·8	—	107·3	—	0·91 ± ·17 (31)	—	1·11 ± ·21	—
El-Amrah and Hou† ...	Late Predynastic	Thomson and MacIver	105·6	119·9	107·6	86·6	1·54 ± ·25 (14)	3·29 ± ·25 (14)	1·15 ± ·19	3·33 ± ·26
Abydos, El-Amrah and Hou†	1st Dyn. (Private Tombs)	,,	33·6	54·9	107·6	86·6	1·97 ± ·25 (14)	1·59 ± ·25 (14)	3·84 ± ·50	2·37 ± ·38
Abyssinia (Tigre District)‡	Modern	S. Sergi	62·0	—	105·2	—	3·91 ± ·18 (28)	—	5·01 ± ·23	—
Abydos, El-Amrah and Hou†	Early Predynastic	Thomson and MacIver	40·9	—	107·6	—	3·35 ± ·25 (14)	—	5·65 ± ·43	—
Hou and Abydos†	12th—15th Dyns.	,,	65·9	87·3	107·6	86·6	4·72 ± ·25 (14)	8·35 ± ·25 (14)	5·78 ± ·31	9·60 ± ·29
Denderah†	6th—12th Dyns.	,,	168·6	140·4	107·6	86·6	7·68 ± ·25 (14)	7·09 ± ·25 (14)	5·85 ± ·19	6·61 ± ·24
El-Kubanieh North‡ ...	Middle Dyns.	Toldt	32·5	—	108·5	—	3·16 ± ·18 (28)	—	6·27 ± ·36	—
Denderah†	Ptolemaic	Thomson and MacIver	76·6	—	107·6	—	6·34 ± ·25 (14)	—	7·09 ± ·29	—
Shekh Ali†	18th Dyn.	,,	40·0	—	107·6	—	4·27 ± ·25 (14)	—	7·33 ± ·44	—
El-Kubanieh South† ...	Early and Middle Dyns.	Toldt	63·6	—	108·5	—	5·89 ± ·18 (28)	—	7·35 ± ·22	—
Badari§	Early Predynastic	Stoessiger	33·8	20·3	107·3	81·7	3·99 ± ·17 (31)	2·13 ± ·17 (31)	7·77 ± ·33	6·55 ± ·52
Thebes†	18th—21st Dyns.	E. Schmidt	161·9	—	101·0	—	9·70 ± ·17 (30)	—	7·77 ± ·14	—
Galla and Somali‖	Modern	Pooled	26·2	—	105·6	—	3·26 ± ·19 (24)	—	7·77 ± ·46	—
Thebes‡	18th—20th Dyns.	Stahr	53·6	—	104·0	—	5·72 ± ·18 (29)	—	8·09 ± ·25	—
Abydos†	18th Dyn.	Thomson and MacIver	49·9	—	107·6	—	6·94 ± ·25 (14)	—	10·17 ± ·37	—
Negroes from Egypt‖ ...	Modern	E. Schmidt	34·1	19·5	102·1	80·5	6·57 ± ·18 (29)	1·17 ± ·18 (29)	12·85 ± ·35	3·73 ± ·56
Denderah†	Roman	Thomson and MacIver	49·3	—	107·6	—	9·34 ± ·25 (14)	—	13·82 ± ·38	—
Gizeh†	26th—30th Dyns.	Pearson and Davin	854·3	—	108·0	—	29·25 ± ·17 (31)	—	15·25 ± ·09	—
Abydos†	1st Dyn.	Motley	32·3	—	107·3	—	7·71 ± ·17 (31)	—	15·52 ± ·34	—
Teita (Kenya Colony)‖ ...	Modern	Kitson	39·8	33·8	107·3	81·7	9·12 ± ·17 (31)	10·07 ± ·17 (31)	15·71 ± ·29	14·89 ± ·25
Sedment‡	9th Dyn.	Woo	37·1	—	107·3	—	8·91 ± ·17 (31)	—	16·14 ± ·31	—
Tamils¶	Modern	Harrower	35·0	—	102·3	—	8·99 ± ·17 (31)	—	17·24 ± ·33	—
Nepalese¶	,,	Morant	45·7	—	107·3	—	11·38 ± ·17 (31)	—	17·76 ± ·27	—
Abydos†	18th and 19th Dyns.	E. Schmidt	29·9	—	102·1	—	8·63 ± ·18 (29)	—	18·66 ± ·38	—
Deshasheh and Medum† ...	4th and 5th Dyns.	Thomson and MacIver	39·9	—	107·6	—	11·38 ± ·25 (14)	—	19·56 ± ·44	—
Dravidians¶	Modern	Turner	30·9	—	103·0	—	10·21 ± ·21 (20)	—	21·46 ± ·45	—
Tanganyika‖	,,	Reid	37·4	18·3	110·6	87·2	12·02 ± ·20 (22)	6·36 ± ·20 (22)	21·50 ± ·36	21·05 ± ·67
Hottentots‖	,,	Pooled	24·2	—	109·1	—	9·98 ± ·19 (24)	—	25·19 ± ·49	—
Hindus¶	,,	,,	74·5	—	104·8	—	26·28 ± ·20 (23)	—	30·18 ± ·23	—
Veddahs¶	,,	,,	33·8	—	105·3	—	17·36 ± ·20 (23)	—	33·91 ± ·39	—

* The $\bar{n}$'s are the mean numbers of skulls available for the characters used in computing the coefficients. The numbers in round brackets following the coefficients are the numbers of characters on which they are based.

† References and male means for these series are given by Morant in *Biometrika*, Vol. XVII (1925), pp. 14—43. The female means of Thomson and MacIver's 6th—12th and 12th—15th Dynasty series are in *Biometrika*, Vol. XXII (1930), p. 70, and the other female means of their series used have not been published.

‡ References and means for these series are given by Woo in *Biometrika*, Vol. XXII (1930), pp. 69 and 74, and in Appendix I to his paper.

§ *Biometrika*, Vol. XIX (1927), Table II, facing p. 117.

‖ References and means for these series are given by Kitson in *Biometrika*, Vol. XXIII (1931), pp. 287—294.

¶ References and means, or the places in *Biometrika* where the means may be found, for these series are given by Woo and Morant in Vol. XXIV (1932), pp. 114—119.

found to be equally removed from those of Early Predynastic and Early Dynastic populations. It is also approximately equally, but more distantly, removed from the earliest Predynastic type (Badari) and from several found in Upper Egypt about the Eighteenth Dynasty. The relationships of the Kerma to most of the Later Dynastic series, including the Roman, are more distant still. Exceptions to any orderly arrangement suggested by these facts must, of course, be expected. They are found, for example, in the rather close connection between the Kerma series and the Ptolemaic and in the dissimilarity of the former to the First Dynasty series from Abydos measured by Motley, to the Ninth Dynasty series from Sedment and to the Fourth and Fifth Dynasty series from Deshasheh and Medum, though it must be remembered that the last series comes from Lower Egypt. In these cases the other series, and not the Kerma, must be supposed peculiar, as has been pointed out in earlier papers. The evidence provides abundant justification for considering that our series represents the typical population found in Upper Egypt in Late Predynastic times, and it can only just be distinguished from other series representing that population. But actually our series is of Twelfth and Thirteenth Dynasty date, though it is clearly differentiated from the contemporary population of Upper Egypt. Obvious conclusions are suggested by these facts. It may be supposed that the colony at Kerma was founded in Late Predynastic times by a body of emigrants from Upper Egypt who were racially typical of the population there and that this type persisted at Kerma unchanged by admixture with any non-Egyptian elements, or by the factors which were modifying the parent population, until the Twelfth and Thirteenth Dynasties. It appears to be extremely probable that these hypotheses are the correct ones, but they could only be submitted to direct proof if we had skeletal evidence of the racial constitution of the population at Kerma between Early Predynastic and Twelfth Dynasty times.

It is certainly unlikely that any coefficients of racial likeness of the same order as the lowest in Table III could be found between the Kerma and any non-Egyptian series. There is a close connection shown with Abyssinians from the Tigre district, but all the closest connections as yet found for this series have been with Dynastic Egyptian types and it is supposed to represent a survival until modern times of part of that population. It has been shown by Stoessiger* that there is a moderately close resemblance between the Predynastic Egyptian series and modern Indian ones, and by Kitson† that the former also resemble various negroid and Bantu negro East African types. It is among these two groups of races that we should expect to find those outside Egypt which will bear the closest resemblance to the Kerma type. Male coefficients of racial likeness were therefore calculated between the Kerma and all the best series available for the two groups and these are given in Table III. Of all the alien types compared, the Galla and Somali shows the lowest reduced coefficient, though it is higher than eleven of the values found with the Egyptian (including the Abyssinian) series. A moderately close connection with that "Hamitic" series was to be expected. Neither is it surprising to find that the

* *Biometrika*, Vol. XIX (1927), pp. 125—135. † *Ibid.* Vol. XXIII (1931), pp. 285—300.

‖ References and means for these series are given by Kitson in *Biometrika*, Vol. XXIII (1931), pp. 287—294.

¶ References and means, or the places in *Biometrika* where the means may be found, for these series are given by Woo and Morant in Vol. XXIV (1932), pp. 114—119.

next closest connection is with a modern series from Egypt of which the origin is unknown and which is only judged to be one of negroes on account of the cranial measurements. The Teita from Kenya Colony come next in order and they are undoubtedly Bantu negroes, while the Tamils are a little further removed. The resemblances in these cases, and those between the Kerma and the other East African and Indian types, are very much more distant than the majority of those between the Kerma and the Egyptian types. The fact that a negro series shows a lower reduced coefficient than three Egyptian series may be attributed to the fact that the latter are not of pure Egyptian origin. It has been shown by Woo, for example, that his Sedment series and that of the Fourth and Fifth Dynasties from Deshasheh and Medum are more closely related to modern Egyptians and modern Cretans than are the other Dynastic Egyptian series. There is again no suggestion that the Kerma population was of any but pure Egyptian origin.

From the 31 series for which male coefficients of racial likeness were calculated, eight were selected for the purpose of comparing the female means with the Kerma values. There are no adequate female means available for most of the other series. If the male and female means for each of two series compared really represent the same population, then we should expect to find that the reduced coefficients for the two sexes do not differ significantly. This is actually the case in five out of the eight comparisons made in Table III, the differences for these being all less than 2·5 times their probable errors. For the other three—viz. the Late Predynastic series from El-Amrah and Hou, the Twelfth to Fifteenth Dynasty from Hou and Abydos, and the modern series of negroes from Egypt—the sexual differences between the coefficients are markedly significant. The means for the last series are based on such small numbers that no stress need be placed on the discordance. In the case of each of the other two a direct comparison of the male and female means suggests that the samples may not represent exactly the same population. It has been shown in section (2) above that the Kerma male and female indices and angles only differ clearly in an unexpected way in the case of the nasal indices and of the angles of the fundamental triangle, the female type having the higher index and the greater nasal angle, indicating a greater degree of prognathism. This might suggest that the female Kerma sample was more negroid than the male, but the coefficients with the Teita and Tanganyika negro series fail entirely to substantiate this view. The Kerma sexual differences for these characters, though of an order which must be considered significant, may only have been due to chance causes.

A detailed comparison of measurements considered singly need not be made. If only the 31 characters used in computing the coefficients of racial likeness are considered, the α's found show whether the differences between the means compared should be considered significant or not. We will suppose, as usual, that a significant difference is indicated if the α is greater than ten. The proportions of significant to non-significant α's are found to differ markedly for different characters as has been observed in all previous comparisons of this kind. The coefficients (given in Table III) were computed between the Kerma male series, on the one hand, and

21 series of the ancient Egyptian type (including the modern Abyssinians), on the other. There are eleven comparisons between median sagittal arcs (S) from nasion to opisthion—the measurement not being available for the other ten series—and not a single one of these is significant. For the basio-bregmatic height (H') only one significant difference out of 21 is found, and for the glabella-occipital length there are only three α's greater than ten in 21 comparisons. Such characters will clearly be of no value if a classification of the material on the basis of single measurements, or groups of a few measurements, is attempted. The only measurements which are likely to be of any value for such purposes are those for which more than 30 per cent., say, of the possible comparisons show significant differences. In the present case these are: 100 B/L (61·9 °/$_\circ$), 100 NB/NH (52·4 °/$_\circ$), B (42·9 °/$_\circ$), 100 G_2/G_1 (42·9 °/$_\circ$), NH (38·1 °/$_\circ$), B' (36·4 °/$_\circ$) and Q' (33·3 °/$_\circ$). The percentage for 100 G_2/G_1 is only based on a total of seven comparisons and no stress can be laid on it, while the other characters selected in this way really only indicate that two primary factors are concerned. The differences between the maximum calvarial breadths (B) are necessarily associated with those between the transverse arcs (Q') and cephalic indices—the calvarial length and height remaining practically constant—and with those between the minimum frontal diameter (B'); while the differences between the nasal heights are necessarily associated with differences between the nasal index—the nasal breadth remaining practically constant. In comparing the Egyptian series Morant found that the essential differences were, in general, those between the calvarial breadths and the characters dependent on that character, and also those between the nasal indices. The Kerma series thus differs most markedly from the Egyptian series in precisely the same characters as they, on the whole, differ most markedly from one another, or, in other words, the Kerma type must be considered to be a perfectly typical representative of the Egyptian stock, as we have previously concluded from a comparison of the coefficients of racial likeness. When comparison is made between the Kerma and the non-Egyptian series the incidence of characters which differ most essentially alters as we pass from series to series and a detailed description of these differences would not be profitable.

(5) *Type Contours.* The individual contours were drawn and the type contours were constructed from them by following the usual methods employed in the Biometric Laboratory. The mean measurements are given in Tables IV—VI and the male and female types are shown in Figs. II—VII. Comparison of the type contour with the mean calliper measurements in Table VII shows a very satisfactory agreement. The differences are all less than 1 mm. except in the case of the male and female S_1'—the chord from nasion to bregma—and the female GL. The difference of 1·4 mm. in the last case is chiefly due to the fact that the two means are based on very different numbers and when found for the same 33 skulls it is reduced to 0·6. It is probable that the rather large differences found in the case of S_1' were caused by the fact that the nasion was not located in precisely the same position when drawing the contours as when taking the calliper length. Table VII shows clearly that the practice of raising or lowering the pointer of the tracer in order

TABLE IV.

Mean Measurements of Kerma Transverse Contours.

Sex	*MA*	1*R*=1*L*	2*R*	2*L*	3*R*	3*L*	4*R*	4*L*	5*R*	5*L*	6*R*	6*L*	7*R*
♂	110·7 (124)	55·4 (124)	58·8 (124)	58·2 (124)	61·4 (120)	61·0 (121)	63·8 (116)	63·2 (121)	64·8 (118)	64·0 (122)	63·8 (121)	63·8 (124)	62·9 (121)
♀	107·7 (90)	52·8 (90)	56·1 (90)	56·5 (90)	58·9 (87)	59·4 (89)	61·8 (84)	61·6 (90)	63·3 (85)	63·0 (90)	63·1 (88)	61·8 (90)	61·6 (89)

Sex	7*L*	8*R*	8*L*	9*R*	9*L*	10*R*	10*L*	$A\frac{1}{4}R$	$A\frac{1}{4}L$	*ZR, R*		*ZR, L*	
										x	*y*	*x*	*y*
♂	62·4 (124)	59·0 (121)	58·4 (124)	50·6 (122)	49·8 (124)	35·9 (124)	35·1 (124)	17·4 (123)	17·1 (124)	3·4 (124)	58·7 (124)	3·2 (124)	58·9 (124)
♀	60·2 (90)	58·4 (89)	56·1 (90)	50·5 (90)	48·0 (90)	36·8 (90)	34·6 (90)	17·4 (90)	16·5 (90)	3·7 (89)	55·8 (89)	4·0 (89)	55·7 (89)

TABLE V.

Mean Measurements of Kerma Horizontal Contours.

Sex	*FO*	$F\frac{1}{4}R$	$F\frac{1}{4}L$	$F\frac{1}{2}R$	$F\frac{1}{2}L$	2*R*	2*L*	$2\frac{1}{2}R$	$2\frac{1}{2}L$	3*R*	3*L*
♂	185·1 (124)	23·1 (124)	21·6 (123)	35·4 (124)	35·0 (123)	47·2 (124)	46·9 (123)	46·1 (124)	46·1 (123)	47·6 (119)	47·4 (124)
♀	174·7 (90)	21·7 (90)	22·3 (90)	32·5 (89)	33·3 (90)	45·1 (90)	44·2 (90)	46·4 (88)	44·7 (89)	48·3 (87)	46·6 (90)

Sex	4*R*	4*L*	5*R*	5*L*	6*R*	6*L*	7*R*	7*L*	8*R*	8*L*
♂	54·2 (117)	53·5 (122)	61·0 (119)	59·6 (123)	65·3 (120)	63·9 (123)	66·3 (123)	65·2 (124)	63·8 (124)	63·1 (124)
♀	53·1 (86)	51·8 (88)	59·2 (87)	57·8 (88)	63·4 (87)	62·3 (90)	64·7 (90)	64·3 (90)	62·3 (90)	62·6 (90)

Sex	9*R*	9*L*	10*R*	10*L*	$O\frac{1}{4}R$	$O\frac{1}{4}L$	*TR*		*TL*	
							x	*y*	*x*	*y*
♂	57·5 (124)	57·1 (124)	45·0 (124)	44·3 (124)	25·9 (124)	25·0 (124)	20·5 (124)	48·9 (124)	19·9 (123)	48·4 (123)
♀	55·5 (90)	55·8 (90)	43·6 (90)	43·7 (90)	24·7 (90)	23·9 (90)	18·6 (89)	46·8 (89)	17·5 (90)	45·2 (90)

TABLE VI.

Mean Measurements of Kerma Median Sagittal Contours.

Sex	$N\gamma$	Ordinates above $N\gamma$												
		$O=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{2}$
♂	182·5 (98)	19·4 (98)	36·5 (98)	58·5 (98)	71·2 (98)	78·7 (98)	83·1 (98)	85·0 (98)	85·3 (98)	81·9 (98)	73·3 (98)	55·1 (98)	27·0 (98)	20·5 (98)
♀	174·2 (73)	17·0 (73)	37·5 (73)	56·7 (73)	68·7 (73)	75·7 (73)	80·1 (73)	81·7 (73)	81·7 (73)	78·4 (73)	69·9 (73)	51·9 (73)	26·3 (73)	19·4 (73)

Sex	Ordinates below $N\gamma$							Vertex		Bregma		Glabella	
	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from N	y	x from N	y	x from N	y
♂	61·5 (75)	53·4 (82)	52·1 (72)	49·0 (96)	38·7 (98)	28·3 (98)	20·2 (98)	100·6 (98)	86·0 (98)	76·8 (98)	83·6 (98)	1·7 (98)	12·1 (98)
♀	54·7 (64)	49·8 (66)	49·3 (58)	47·5 (73)	36·4 (73)	27·8 (73)	20·6 (73)	94·9 (73)	82·4 (73)	73·0 (73)	80·5 (73)	1·2 (73)	12·7 (73)

Sex	Occipital Pt.		λ		Sub-Orb.		Aur. Pt.		Opisthion		Inion		Basion	
	x from γ	y*	x from γ	y	x from N	y	x from γ	y	x from γ	y	x from γ	y	from γ	from N
♂	0·4 (98)	3·1 (98)	5·2 (98)	27·8 (98)	8·3 (91)	27·8 (91)	93·1 (98)	26·7 (98)	57·4 (92)	54·4 (92)	15·8 (85)	37·8 (85)	106·4 (84)	102·0 (84)
♀	0·3 (73)	1·5 (73)	4·7 (73)	26·9 (73)	6·5 (71)	25·8 (71)	90·2 (73)	26·3 (73)	55·7 (69)	52·9 (69)	17·0 (40)	36·0 (40)	101·5 (64)	96·5 (64)

Sex	Alv. Pt.		Nose					Palate				Frontal	
	from N	from Bas.	(i)	(ii)	(iii)	$\angle LN\gamma$	NL	P'		P		Max. Sub. to $N\beta$	
								x from N	y	x from Alv.	y	x from N	y
♂	69·6 (50)	96·2 (46)	0·4 (80)	1·9 (73)	5·2 (48)	116°·9 (17)	19·4 (17)	47·4 (49)	50·6 (49)	37·0 (41)	16·4 (41)	49·7 (98)	26·4 (98)
♀	66·4 (37)	95·1 (33)	1·0 (63)	2·9 (59)	6·4 (40)	117°·7 (14)	19·4 (14)	43·4 (60)	47·6 (60)	37·3 (36)	15·3 (36)	47·0 (73)	26·2 (73)

Sex	Occipital Max. Sub. to λ Op.		Max. Sub. to $N\lambda$		Max. Sub. to GI		Sp.		Sub. from ½ Bas.—Sp. Chord		N.S.		Prosthion	
	x from λ	y	x from N	y	x from G	y	x from N	y	x from Bas.	y	x from N	y	x from N	y
♂	50·2 (93)	27·4 (93)	90·4 (98)	71·4 (98)	104·0 (85)	105·0 (85)	68·6 (82)	34·7 (82)	14·2 (81)	1·2 (81)	3·8 (45)	49·3 (45)	7·8 (52)	65·7 (52)
♀	45·8 (69)	26·3 (69)	84·3 (73)	68·8 (73)	95·1 (39)	98·3 (39)	64·3 (55)	33·1 (55)	12·3 (59)	1·5 (59)	7·7 (57)	45·2 (57)	10·6 (44)	60·8 (44)

* The occipital points are above the $N\gamma$ lines.

TABLE VII.

A Comparison of Calliper and Type Contour Measurements.*

Character	Male		Female	
	Contour	Calliper	Contour	Calliper
L	185·2 (98)	185·2 (138)	176·7 (73)	177·2 (114)
OH	110·7 (124)	110·7 (125)	107·7 (90)	107·4 (91)
S_1'	113·5 (98)	112·1 (126)†	108·7 (73)	107·4 (109)‡
S_2'	114·9 (98)	115·1 (128)	110·4 (73)	110·4 (106)
S_3'	97·4 (92)	97·2 (124)	94·7 (69)	94·3 (97)
fml	36·6 (84)	36·4 (112)	34·7 (64)	34·6 (92)
H'	134·3 (84)	133·4 (93)	128·1 (64)	127·8 (88)
G'H	69·6 (50)	69·6 (111)	66·4 (37)	65·7 (82)
GL	96·2 (46)	96·5 (97)	95·1 (33)	93·7 (68)§
LB	102·0 (84)	101·8 (117)	96·5 (64)	96·6 (89)

* The contour *OH* is the height (*MA*) of the transverse type contour, and all other contour measurements are taken from the sagittal type. The maximum length (*L*) is measured on that figure; *G'H*, *GL* and *LB* are used in its construction and all other measurements compared are found by calculation from lengths used in the construction of the type.

† For the 98 male skulls for which sagittal contours were drawn the mean calliper S_1' is 112·1.

‡ For the 73 female skulls for which sagittal contours were drawn the mean calliper S_1' is 107·3.

§ For the 33 female skulls for which the contour *GL* is given the mean calliper *GL* is 94·5.

to pass exactly through the "points" in drawing the sagittal figures does not introduce any serious discrepancies. For all practical purposes there is little danger in assuming that all the points shown on that section really lie in a single plane in the case of each individual skull*. If the series be a long one it might be more satisfactory to omit the specimens for which this is least true.

The male transverse type is almost exactly symmetrical, all differences between the right and left sides of the same parallel being less than 1 mm. Greater differences are found in the case of the female figure and the largest is 2·5 for the ninth parallel. The Kerma sections show no striking peculiarities. A number of indices, providing measurements of shape, have been derived from these figures for the purpose of making racial comparisons more exact, and for all these the Kerma values fall well within the ranges given by racial series previously studied. The male and female indices are closely similar as is generally found. By superposition, with aid of the tracings provided, the Kerma outlines are found to be extraordinarily close of those provided by the Teita negro series and rather further removed from the Badari types.

The horizontal type contours are very approximately symmetrical, the maximum difference between the right and left sides of the same parallel being 1·5 in the case of the male figure and 1·7 for the female. They are again found to have no

* [I do not agree with this view. The "practice" does not seem to me satisfactory, as the sagittal section of the individual skull ceases to be a section of that skull at all, and the diagraph thus used can lose the advantages it has over the dioptograph. Ed.]

unusual characters and the indices which have been used for comparative purposes are all, with one exception, within the ranges previously found. The most striking sexual difference in the case of the horizontal contours is seen in the shapes of the sections of the temporal fossae. Immediately behind the temporal lines (marked by the points *TR* and *TL*) the male outline is contracted to a greater extent than the female. This difference has been measured by expressing the total length of

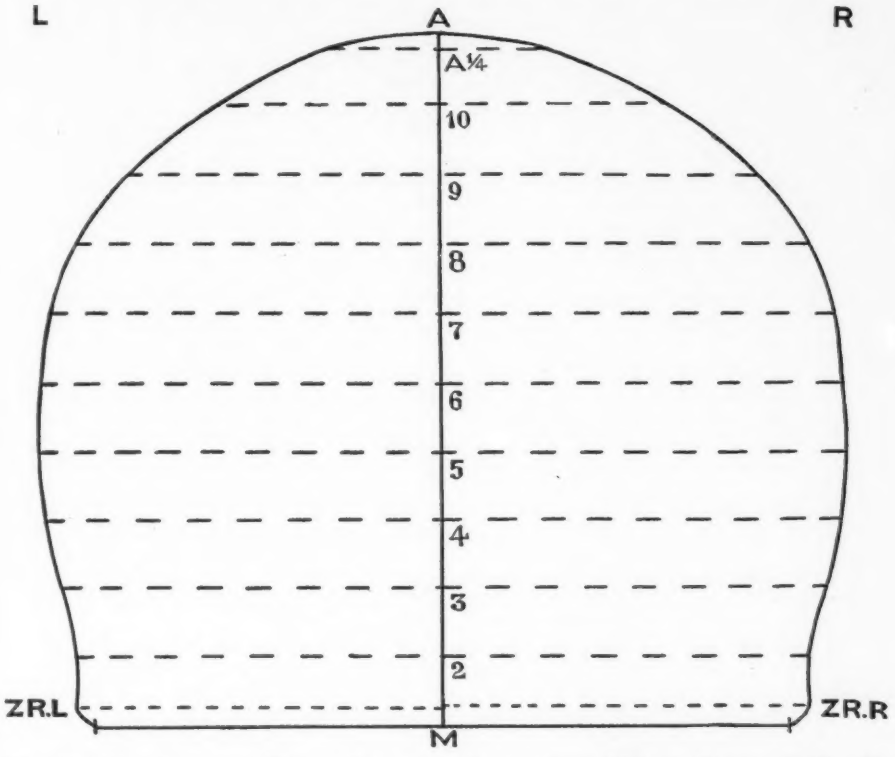


Fig. II Transverse Type Contour, based on 124 ♂ Kerma Skulls.

parallel 3 as a percentage of the width between the temporal lines, i.e. very approximately $TR(y) + TL(y)$. The male Kerma index is 97·7 and the female 103·1, and this difference is rather greater than any previously found between the male and female values for the same racial series. The male index is also lower, indicating more marked temporal fossae, than any previously found. The Kerma type contours are again found to be remarkably similar to the Teita, the only difference which appears to have any significance being dependent on the fact that

the sections of the temporal fossae are more accentuated on the Kerma male figure. The Kerma and Badari types differ more markedly.

The sagittal type sections possess no features which would have been unexpected in contours representing an Egyptian series, except, perhaps, a rather greater degree of prognathism. All the indices and angles which have been used to compare the most salient racial differences of the median sagittal section have values for the

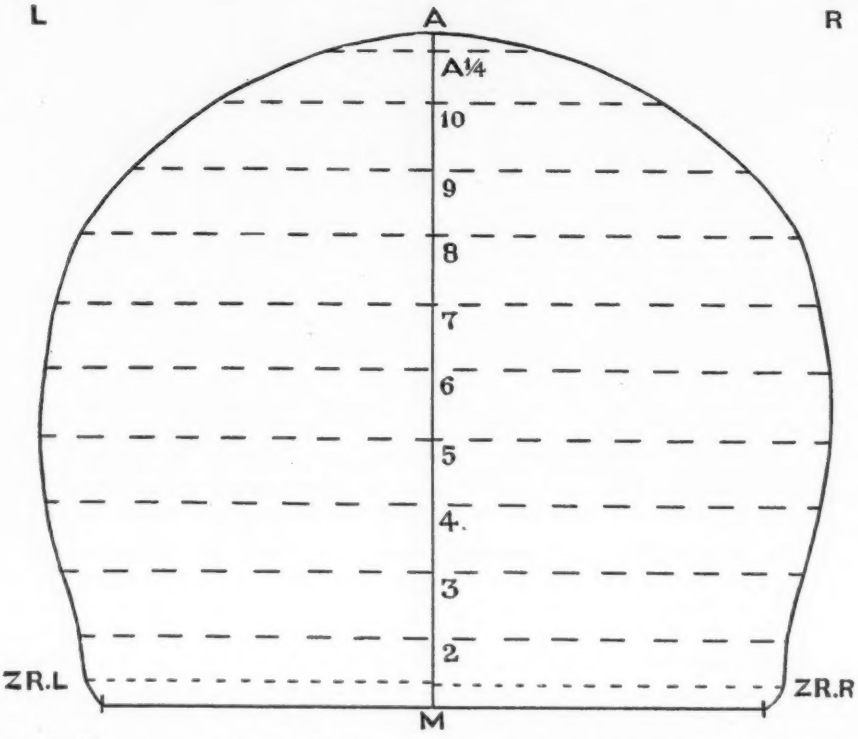


Fig. III Transverse Type Contour, based on 90 ♀ Kerma Skulls.

Kerma series which fall well within the ranges given by the types previously recorded. Mean sexual differences are generally found for some of these and those of the same sign and the same order are observed in the present series. For example, the index expressing the maximum subtense of the frontal arc as a percentage of the $N\beta$ chord is 23·2 for the males and 24·1 for the females, and the fact that the female frontal bone is, on the average, more vertical than the male can be illustrated by angular measurements. A comparison of the mean indices

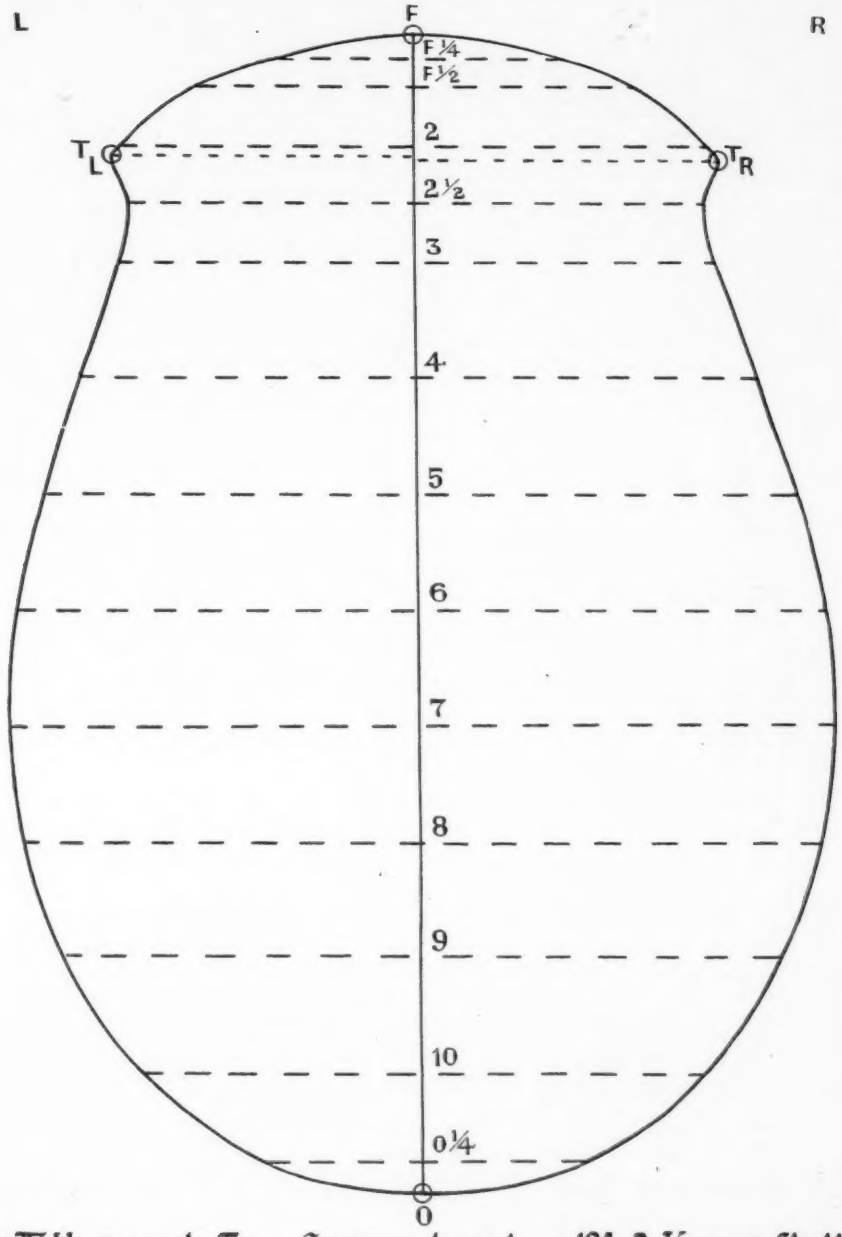


Fig. IV Horizontal Type Contour, based on 124 ♂ Kerma Skulls.

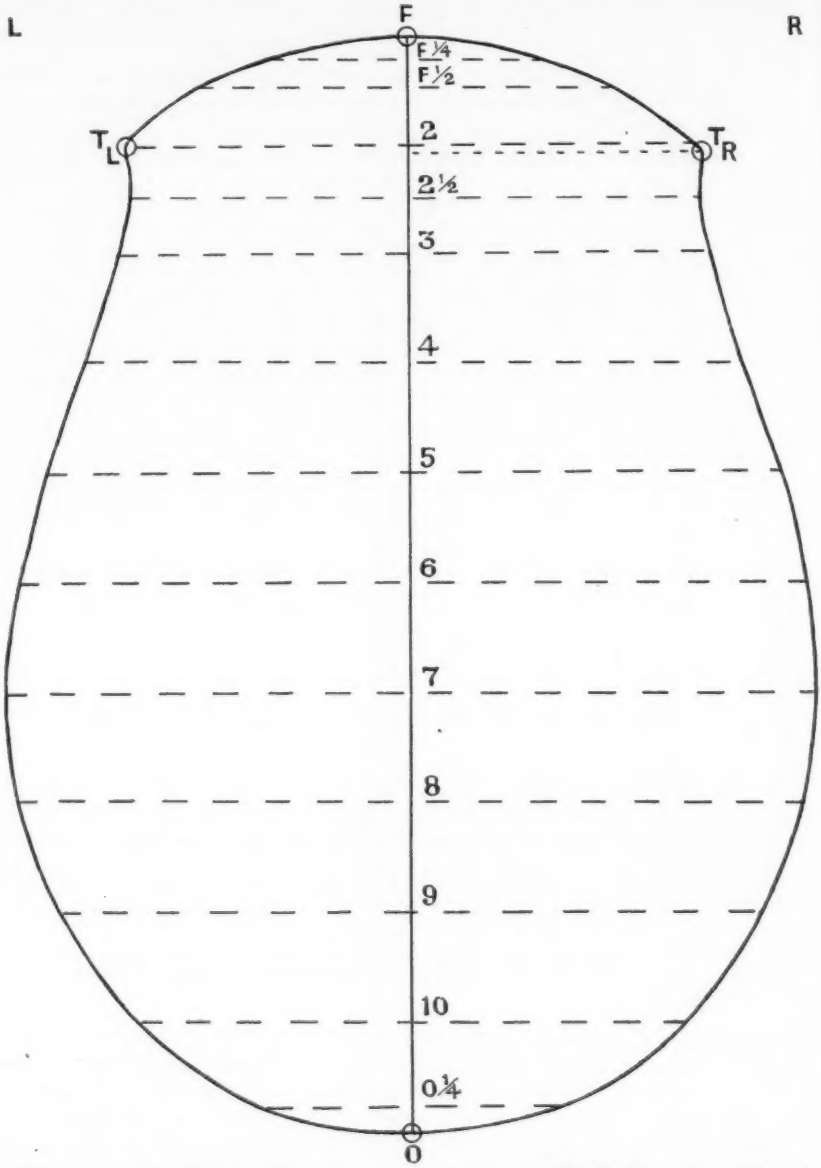


Fig.V Horizontal Type Contour, based on 90 ♀ Kerma Skulls.

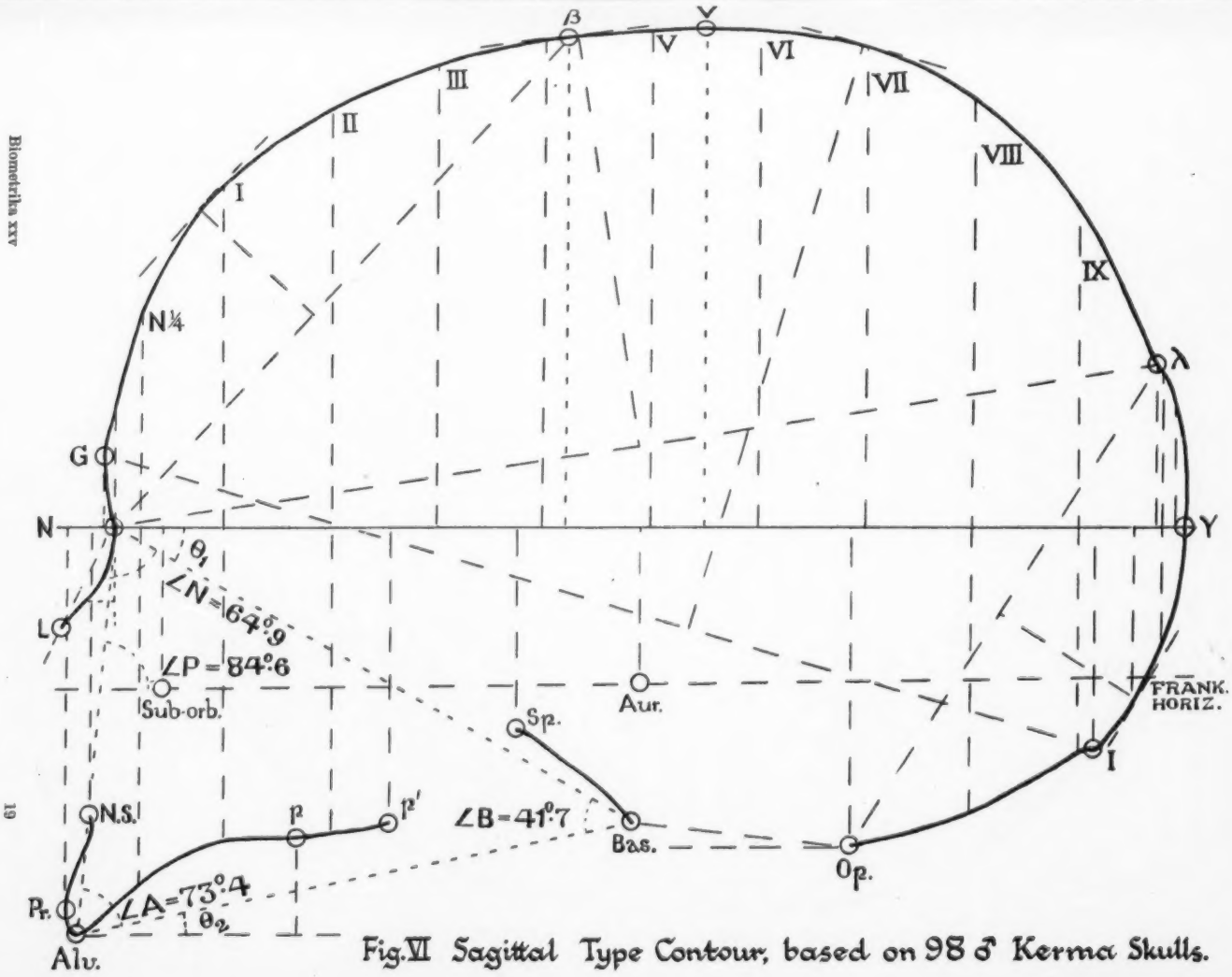


Fig. VI Sagittal Type Contour, based on 98 ♂ Kerma Skulls.

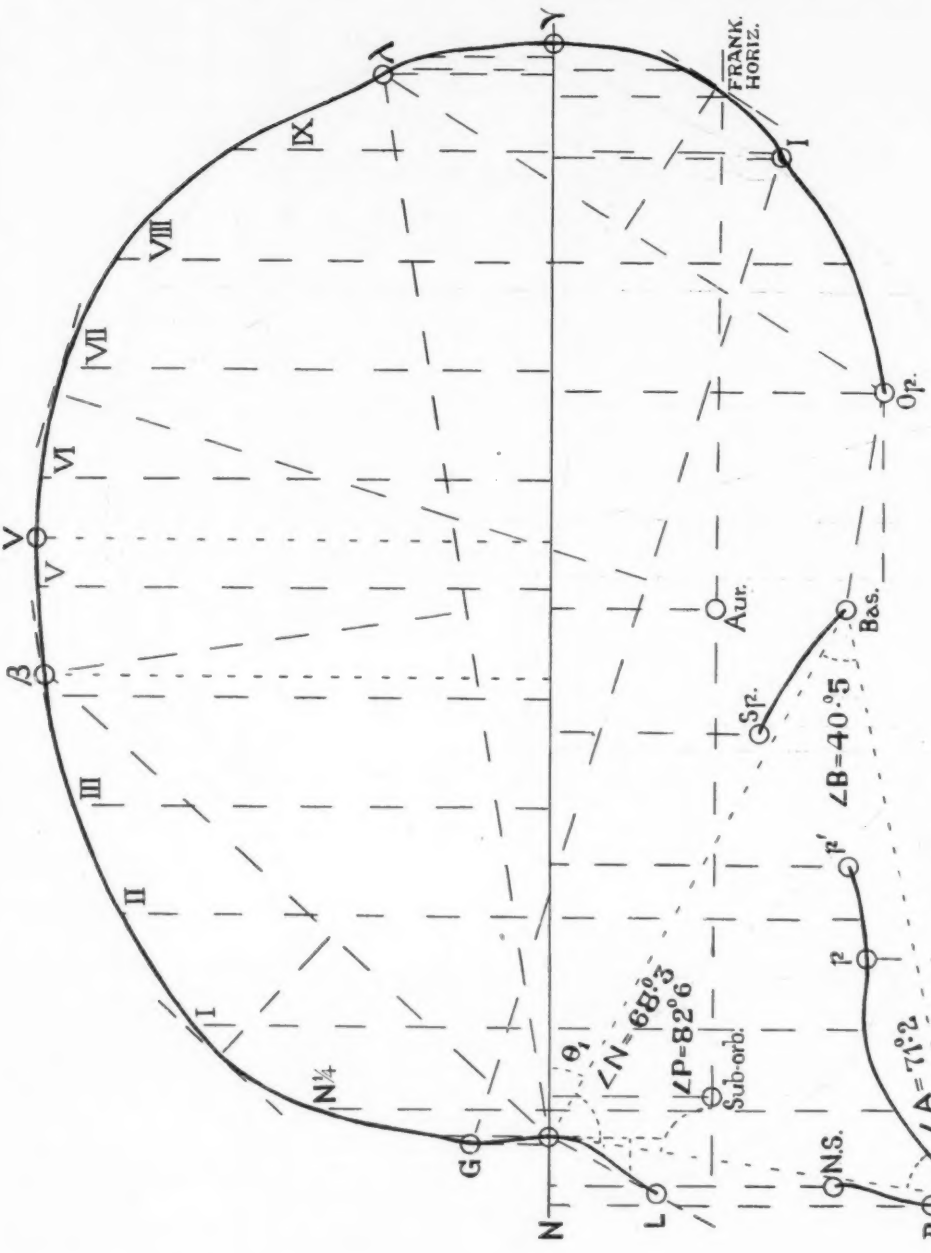


Fig. VII Sagittal Type Contour, based on 73 ♀ Kerma Skulls.

and angles derived from calliper measurements for the two series showed that the most marked differences in shape which would not have been anticipated were found for the angles of the fundamental triangle. Confirmation of the correctness of the means can be obtained from the sagittal type contours. The values are:

Sex	Method of Measurement	$N\angle$	$A\angle$	$B\angle$
♂	By callipers on skull	64°·9 (95)	74°·1 (95)	41°·0 (95)
	Scaled on type contour	64°·9 (46)	73°·4 (46)	41°·7 (46)
♀	By callipers on skull	67°·6 (63)	72°·1 (63)	40°·3 (63)
	Scaled on type contour	68°·3 (33)	71°·2 (33)	40°·5 (33)

By superposing the types it is found that the Kerma sagittal section bears a much closer resemblance to the Badari than to the Teita. Strangely enough, considering that one of these series represents a tribe of Bantu negroes, the degrees of prognathism of these three are almost identical, but the section of the nasal bones projects considerably less in the Teita than in the Egyptian types.

(6) *Conclusions.* The series of skulls described in this paper represents the population of the Egyptian settlement at Kerma (Nubia) in the Twelfth and Thirteenth Dynasties. Only adults were dealt with and of these 141 appeared to be males and 114 females. No distinction can be made between the series representing the Twelfth and Thirteenth Dynasties respectively. The male and female series may be supposed to represent the same racial type. The male variabilities are rather greater than the female, as is generally found, and the population as a whole must be considered rather more variable than those in some Dynastic Eygyptian cemeteries, but still as homogeneous as most for which cranial samples are available. The type is distinctly negroid and the frequencies with which some anomalies are found assign it to an intermediate position between those of Bantu negroes on the one hand and European types on the other. The coefficients of racial likeness show that all the closest relationships of the Kerma series are with Upper Egyptian series of Predynastic and Early Dynastic date. The closest connections found are with Late Predynastic types and the Kerma sample may be supposed to represent the typical population of Upper Egypt at that time. It is distinctly further removed from the types contemporary with it found in Upper Egypt. Hence it is concluded that the settlement was probably founded in Late Predynastic times and that the racial type there persisted unchanged until the Thirteenth Dynasty although the parent population had been modified in the interval. Comparisons are also made with negro and Indian series, but no unexpectedly close connections are shown. The type contours are provided.

In conclusion I must thank Mr G. C. Dunning, some of whose measurements I have used, Miss M. Kirby, who drew the map and type contours, and Dr G. M. Morant, to whom I am greatly indebted for much help and encouragement.

Fig. VII Sagittal Type Contour, based on 73 ♀ Kerma Skulls.

DESCRIPTION OF PLATES OF KERMA SKULLS.

I. Typical male skull (No. 27), *Norma facialis* (0·9 natural size).

II. Typical male skull (No. 27), *Norma lateralis* (0·6 natural size).

III. (*a*) Typical male skull (No. 27), *Norma verticalis* (0·7 natural size).
This specimen has a cephalic index of 72·0, and the male mean for the series is 72·2.

(*b*) Male skull (No. 130; 0·7 natural size). This shows holes presumed to have been made by insects after death.

IV. (*a*) Male skull (No. 74; 0·7 natural size), with a depressed wound on the parietal and temporal bones and a healed fracture of the malar bone.

(*b*) Female skull (No. 153; 1·2 natural size), with large Wormian bones between the temporal squama and parietal bone.

V. (*a*) Female skull (No. 274; 0·7 natural size), showing diseased areas on either side of the lambda.

(*b*) Male skull (No. 36; 1·4 natural size), showing constriction of the basi-occipital where it unites with the sphenoid.

(*c*) Male skull (No. 64; 0·8 natural size), showing a wound below the right orbit.

(*d*) Female skull (No. 216; 1·3 natural size), showing failure of the tympanic elements to unite.

VI. (*a*) Male skull (No. 92; 1·2 natural size), with the sockets of two teeth other than molars lacking and a diseased opening on the left-hand side.

(*b*) Female skull (No. 216; 1·2 natural size), with no sockets for lateral incisors.

(*c*) Female skull (No. 155; 1·3 natural size), with no sockets for the left canine and right third molar.

(*d*) Male skull (No. 107; 1·5 natural size), with sockets for three incisors only, the socket for the central incisor being apparently between the pre-maxillary bones.

Serial No.	Grave Mark	Probable Dynasty	*C*	*B'*	*B''*	*B*	*Bias-terionic B*	*F*	*L*	*H'*	*H*	*OH*	*LB*	S_1'
1	XB/JB	XIII	1451·5	95·0	—	138·5	109·3	184·5	186·5	137	139·5	113·5	102·2	115·2
2	XB/WC	„	1534·5	98·5	—	137	112·5	196·5	196	141	143	114	106·0	116·8
3	XB/WA	„	—	91·8	—	—	106·0	179	181·5	134·5	—	—	—	112·8
4	XB/O	„	—	98·3	111·5	125	—	189·5	189	—	—	119	—	118·1
5	XB/WH	„	—	95·2	—	128·5	102·1	181·5	183	132·5	134·5	112·5	101·3	110·0
6	XB/MG	„	—	93·2	114	132	106·0	188·5	190	—	138·5	111·5	104·0	116·0
7	XB/BC	„	—	92·0	—	132	105·8	186·5	184	138	139·5	116·5	94·6	115·6
8	XB/ZA	„	1571·5	88·0	109?	138·5	113·2	195	197	—	148	122	104·0	—
9	XB/OC	„	—	91·8	—	126	101·7	198	198	—	—	113	—	—
10	XB/XC (B)	„	—	96·0	—	134	—	190	191·5	—	—	109	—	114·5
11	XB/ZC (B)	„	—	88·5	—	133·5	110·6	188	188·5	—	—	114·5	—	114·5
13	XB/ZK	„	1409·5	87·0	113	130	110·0	182	183·5	133·5	136	111	103·4	111·8
14	XB/YA	„	1204	88·2	—	134	105·0	175	176	134	135·5	109·5	101·5	111·4
15	XB/ZC	„	—	95·5	—	140	109·6?	195·5	197·5	132·5	—	110·5	107·0	118·2
16	XB/MA (i)	„	1454·5	95·2	117	137·5	109·9	183	185	137·5	140	118	99·5	117·6
17	XB/RC	„	—	90·4	—	130·5	106·3	187·5	190	135	—	—	—	113·3
18	XB/LB	„	—	91·8	111·5	124	96·0	172	177	135?	—	—	101·5	113·8
19	XB/MA (ii)	„	—	93·0	—	138	112·9?	186·5	187·5	140	141·5	114·5	102·2	115·4
20	XB/IC	„	1309	95·2	—	130	101·2	182	184·5	—	136	109·5	104·0	—
21	XB/XL	„	1412·5	91·2	—	130	105·2	190·5	191	134·5	138	109	103·2	111·1
22	XB/TL	„	—	92·6	—	133	101·0	171·5	175·5	130	—	—	—	102·3
23	1041/B	„	—	99·1	—	136·5	—	182·5	186·5	—	133	105·5	110·2	—
24	437 B	XII	—	92·2	—	126·5	108·0?	187·5	190	—	—	—	101·3	115·0
25	1026/B	XIII	—	87·7	—	135·5	105·6?	187·5	187	145	146	119	106·3	117·5
26	330 B	XII	1315	91·0	109·5	135·5	100·0?	178·5	180	131	135	112·5	96·0	110·7
27	451 B	„	1312	95·0	111·5	135	100·2	185	187·5	—	132	108·5	102·3	112·6
28	446 B	„	1351·5	88·0	107·5?	126	103·6	184	186	136	139	109·5	107·1	112·0
29	1026/B	XIII	1253·5	95·0	106	137	107·0?	180	180	128·5	130·5	109	95·6	112·8
30	1018/B	„	1482	91·5	114	137	110·3	180·5	182·5	139	141·5	114·5	99·2	111·4
31	III 301	XII	—	86·6	—	144·5	—	175	175	132?	—	115	—	107·3
32	III 301 (C)	„	—	95·2	108·5	129	107·4	193	194	142·5	142·5	116	108·5	115·1
33	IV F/FB	„	—	90·3	109?	128·5	109·2	180	182·5	—	—	116	108·0	110·7
34	414 B	„	1371	93·7	115	126	107·5	185·5	186·5	—	136	111·5	103·7	115·1
35	IV B/CB	„	1145	89·8	104·5	135·5	107·6	168	172	126	128	101·5	102·1	105·1
36	III 301 (XA)	„	1342·5	90·7	112	136·5	—	184·5	185	133	134	110	101·2	108·7
37	IV B/C (BA)	„	—	96·6	107?	128·5	106·3?	196·5	198	142	145·5	122·5	111·6	116·3
38	IV B/AB	„	1632·5	99·5	118	134·5	109·8	196	198	138	140	115	108·2	121·3
39	IV B/JA	„	1368·5	85·8	—	128·5	101·6	181	184·5	134·5	137	107·5	99·0	111·2
40	315	„	—	98·6	—	133·5	—	187·5	189·5	—	—	—	100·8	112·0
41	IV 314?	„	—	91·0	—	134	98·2	179·5	183	124·5	127	101·5	95·7	105·8
43	313	„	1499	93·7	118	137·5	112·4	189	193	134	135	109·5	100·9	114·4
44	427 A	„	—	85·0	—	134·5	100·0	179	181	135·5	138	113·5	101·0	—
45	427	„	1434	100·0	115?	135·5	104·4	181·5	183·5	135	137	110·5	106·9	112·5
46	1041/D	XIII	—	96·2	—	136	104·1	191	191	—	—	111·5	106·5	118·5
47	1024	„	1337	87·9	—	129	—	187	189	—	138	110	102·8	—
48	1024/A	„	1334·5	92·8	—	133	108·9	185	188	—	141	115·5	103·6	113·2
49	1065/A	„	—	96·0	116	131·5	—	185·5	186	130	131·5	104	99·0	116·6
50	1048/A	„	1556	98·4	—	135	—	203·5	205	—	148	121·5	109·4	117·3
51	1048/B	„	—	92·7	—	134·5	113·5	178·5	182	129	—	—	101·8	110·6
52	435 A″	XII	1470·5	99·4	114?	134·5	105·0	187·5	189·5	135	—	114	102·6	119·2
53	1058	XIII	—	92·7	—	127·5	107·4	185·5	186·5	—	135	106	100·3	—
54	455	XII	1326	89·2	112·5	139·5	110·2	183	185	—	133	109·5	98·7	—
55	449	„	1338	98·0	112·5	133	104·3	181	184	131	131	107·5	100·8	112·5
56	340	„	1472·5	90·0	—	138	100·1	184	187·5	137	137·5	112·5	103·2	111·6
57	308 A	„	1221	97·0	111·5	130	105·8	175	178	126·5	129	108	101·1	103·6
58	325	„	—	91·3	—	136?	106·6	198	197	133	135?	115	95·9	—
59	III 305	„	1471·5	101·8	—	135	—	190·5	193	—	146·5	116·5	110·7	121·4
60	420 A′	„	1320	100·8	—	133·5	100·0	188	190·5	123	126·5	103	101·8	108·6
61	K 420	„	—	95·4	—	136	107·3	186·5	187·5	137	138	109·5	100·4	115·1
62	414	„	1434	97·0	119·5	133	—	190	191·5	130·5?	131	109·5	103·9	117·3
63	320 B	„	1484	99·6	113	137·5	114·0	185	189	133	134	110·5	105·9	110·8
64	320 A	„	1325·5	89·3	112·5	137·5	99·5	180·5	183·5	132·5	132·5	109	96·5	116·8
65	III 301 K	„	1352	91·7	—	133·5	105·0?	192·5	193	—	130·5	108	105·0	119·3
66	163 (ii)	XIII	—	99·5	111	133·5	104·9	182·5	183	—	—	113·5	—	113·0
67	452 X/B	XII	1351	86·2	—	131·5	96·1	181'	186	132·5	134·5	109	99·9	113·4
68	1621 C	XIII	—	87·3	—	134	100·4?	—	—	139·5	—	—	—	—
69	1045/C	„	1210·5	103·4	121	135·5	101·6	178	178	132	132	111	101·2	113·8
70	1042/C	„	1419	99·3	115	133	100·2?	189	188·5	135	134·5	111·5	103·8	113·1
71	1805 AC	„	1350·5	89·5	120	136	99·0	182	183	135	136·5	111·5	105·3	107·6
72	1619/B ii	„	1351·5	91·2	120	129	103·7	183	187	131	133	109	98·6	115·4
73	1803 X	„	—	95·8	114·5	139	106·0	191·5	190·5	—	—	122·5	—	124·1

OH	*LB*	S_1'	S_2'	S_3'	S_1	S_2	S_3	*S*	*U*	*Q'*	*G'H*	*GL*	*GB*	*J*	*NH'*	*NH, R*	*NH, L*	*NB*	*DS*
113·5	102·2	115·2	117·8	101·0	130	132	120	382	518	309	71·0	—	94·7	130·5	51·7	52·6	51·6	24·4	—
114	106·0	116·8	124·5	97·6	133	137	118	388	536	313	72·4	96·4	97·8	134	52·0	51·8	51·5	24·2	13·9
—	—	112·8	116·5	93·6	126	130	107	363	—	—	—	—	—	—	—	—	—	—	—
119	—	118·1	115·2	99·1	136	124	118	378	513	309	69·5	—	90·0	—	52·3	52·0	51·9	23·4	14·9
112·5	101·3	110·0	112·8	97·5	125	128	117	370	499	305	67·4	92·9	100·2	—	47·0	47·1	46·8	25·6	—
111·5	104·0	116·0	116·4	99·1	134	130	119	383	519	305	66·8	98·0	—	128	50·8	52·2	51·4	25·2	—
116·5	94·6	115·6	116·7	101·0	136	132	121	389	511	316	67·9	86·6	98·4	—	48·0	46·6	47·3	22·8	—
122	104·0	—	—	99·1	—	—	121	406	536	322	64·4	100·1	94·0	135·5	45·7	47·8	45·4	27·0	—
113	—	—	—	102·6	—	—	124	405	533	303	—	—	87·3	—	—	—	—	—	—
109	—	114·5	116·8	95·9	134	130	119	383	520	304	67·2	—	95·4	128·5	49·0	48·8	47·9	25·8	14·2
114·5	—	114·5	119·0	—	129	133	—	—	522	307	69·9	—	94·0	127·5	50·6	52·8	50·9	25·8	13·8
111	103·4	111·8	121·0	93·6	125	137	108	370	506	297	68·8	105·5	101·9	135	50·2	49·6	50·2	27·7	—
109·5	101·5	111·4	106·8	96·2	130	117	116	363	490	300	61·3	95·9	91·0	123	45·1	45·4	45·9	24·5	—
110·5	107·0	118·2	118·3	99·6	137	129	123	389	536	313	74·3	98·1	93·9	—	50·2	50·3	50·1	22·5	—
118	99·5	117·6	123·8	93·7	138	139	110	387	515	322	66·7	94·5	96·5	129	46·6	45·8	46·6	24·4	11·0
—	—	113·3	119·5	98·3	128	132	116	376	517	—	—	—	—	—	—	—	—	—	—
—	101·5	113·8	105·8	96·1	129	118	111	358	482 ?	—	—	—	—	—	—	—	—	—	—
114·5	102·2	115·4	120·3	108·0	132	136	133	401	522 ?	316	—	—	—	—	—	—	—	—	—
109·5	104·0	—	—	—	—	—	—	391	505	299	70·5	103·4	105·5	133	48·7	50·0	49·4	27·2	—
109	103·2	111·1	108·1	110·0	129	117	137	383	514	300	—	92·5	94·7	122	49·0	49·5	49·0	26·8	—
—	—	102·3	113·0	92·8	119	125	111	353	487	—	—	—	—	—	—	—	—	—	—
105·5	110·2	—	—	—	—	—	—	354	514	306	77·7	104·3	98·4	133·5	56·2	56·0	56·7	24·7	—
—	101·3	115·0	116·7	94·7	131	127	116	374	512 ?	—	—	—	—	—	—	—	—	—	—
119	106·3	117·5	111·6	108·5	134	120	126	380	525	324	72·2	99·2	101·9	—	54·6	54·3	55·0	28·0	—
112·5	96·0	110·7	118·2	93·3	125	133	110	368	501	310	65·6	92·0	92·0	129	46·0	47·5	47·5	27·5	—
108·5	102·3	112·6	114·2	98·5	129	127	120	376	517	304	76·6	96·5	103·1	131·5	54·2	54·0	54·3	26·5	—
109·5	107·1	112·0	113·6	100·7	125	127	116	368	501	296	79·0	95·0	92·0	—	55·5	56·5	56·8	24·0	—
109	95·6	112·8	112·0	98·8	129	127	124	380	498	307	71·9	100·0	100·7	129	52·9	51·8	52·8	24·9	—
114·5	99·2	111·4	122·8	99·9	127	139	116	382	510	314	71·0	97·0	96·0	132	49·6	49·2	50·0	24·5	—
115	—	107·3	113·1	94·7	134	130	111	375	500	317	—	—	90·5	—	—	—	—	25·2	—
116	108·5	115·1	123·6	96·6	133	138	117	388	518	311	—	—	—	121	—	—	—	—	—
116	108·0	110·7	115·5	89·6	124	129	109	362	503	310	74·3	104·6	—	—	—	49·0	49·0	—	—
111·5	103·7	115·1	115·6	92·7	132	128	113	373	508	302	70·6	99·3	90·5	128	52·2	51·8	52·4	26·4	13·2
101·5	102·1	105·1	102·3	94·3	117	115	110	342	481	286	64·8	101·4	100·5	134	50·0	49·2	50·0	27·8	12·7
110	101·2	108·7	116·1	101·0	125	128	126	379	514	302	66·0	89·9	103·1	133	48·7	48·7	48·4	24·5	—
122·5	111·6	116·3	125·4	109·3	131	141	129	401	533	322	63·0	104·3	94·0	—	46·9	46·7	46·6	26·1	—
115	108·2	121·3	121·3	98·5	143	137	121	401	538	316	75·7	102·3	103·3	—	55·2	54·4	54·2	28·2	—
107·5	99·0	111·2	121·1	98·7	126	137	117	380	502	290	65·0	91·5	88·5	124·5	49·5	48·6	49·5	27·9	—
—	100·8	112·0	107·8	111·5	130	116	141	387	524	—	—	—	—	—	—	—	—	—	—
101·5	95·7	105·8	102·7	106·4	120	114	133	367	504	294	77·7	90·0	94·0	133	59·9	59·7	60·0	28·0	—
109·5	100·9	114·4	117·2	101·6	128	134	127	388	531	303	70·6	96·2	90·5	130	50·5	51·8	52·4	26·1	—
113·5	101·0	—	—	—	—	—	—	382	501	309	70·8	92·0	94·0	—	—	—	—	—	—
110·5	106·9	112·5	113·5	96·3	127	125	114	366	513	305	70·8	94·3	90·9	131	52·6	53·4	53·4	25·8	14·3
111·5	106·5	118·5	123·0	97·1	137	138	112	387	517	310	73·0	100·5	97·7	117	50·3	49·0	50·1	26·9	8·7
110	102·8	—	—	—	—	—	—	381	510	302	76·5	101·4	98·2	120	52·5	51·0	52·0	26·0	—
115·5	103·6	113·2	—	—	128	—	—	378	517	312	73·0	102·5	98·0	127	51·4	50·2	51·4	27·4	12·1
104	99·0	116·6	108·0	99·0	136	118	120	374	516	296	71·2	90·5	94·7	126	52·4	54·2	52·8	28·1	9·2
121·5	109·4	117·3	131·1	104·4	133	148	132	413	548	323	75·6	106·8	106·2	135	53·1	52·5	52·4	27·1	—
—	101·8	110·6	115·0	96·3	121	130	113	364	503	—	—	—	—	—	—	—	—	—	—
114	102·6	119·2	105·7	103·9	137	115	128	380	523	318	—	—	93·0	128	—	—	—	—	—
106	100·3	—	—	—	—	—	—	374	508	291	70·1	100·3	96·1	—	48·5	49·2	49·5	25·9	—
109·5	98·7	—	—	—	—	—	—	372	514	302	70·2	97·6	98·6	133	52·5	52·5	51·9	25·6	—
107·5	100·8	112·5	119·5	93·2	130	135	110	375	506	300	72·3	99·8	94·0	129	50·0	49·9	49·6	26·1	—
112·5	103·2	111·6	115·1	104·2	129	127	127	383	513	310	73·0	93·7	92·5	124 ?	51·3	52·0	51·6	23·9	—
108	101·1	103·6	112·6	93·1	118	126	110	354	492 ?	299	67·3	98·5	93·6	127·5	45·0	45·9	44·0	22·8	—
115	95·9	—	—	—	—	—	—	405	540 ?	313	68·7	91·3	—	—	48·9	49·7	49·0	27·0	—
116·5	110·7	121·4	120·0	101·0	138	132	118	388	534	317	73·6	99·4	102·4	141·5	54·5	54·1	54·6	27·4	14·0
103	101·8	108·6	114·6	97·1	124	125	119	368	516	292	70·5	95·4	98·5	124	51·6	51·5	51·9	26·0	—
109·5	100·4	115·1	119·2	99·1	130	136	120	386	499	296	81·0	101·0	—	—	54·3	55·0	56·2	29·3	—
109·5	103·9	117·3	114·3	91·2	138	128	109	375	522	303	66·5	95·7	95·6	126	48·5	49·8	49·3	25·9	9·6
110·5	105·9	110·8	116·5	93·2	125	130	114	369	527	311	77·0	99·5	90·5	127·5	55·3	53·9	55·9	24·6	—
109	96·5	116·8	112·9	94·7	134	126	117	377	503	305	61·4	91·0	—	—	46·1	46·1	47·5	28·4	—
108	105·0	119·3	114·7	89·8	140	126	114	380	523	301	66·8	95·7	92·6	130	48·2	48·6	48·8	23·0	—
113·5	—	113·0	119·7	96·2	127	132	112	371	504	304	65·0	—	89·4	125·5	44·6	45·8	45·8	24·0	8·5
109	99·9	113·4	113·1	95·3	131	128	112	371	505	302	63·0	91·8	92·8	—	48·2	47·7	47·4	25·5	10·5
—	—	—	118·6	97·7	—	137	118	—	—	—	—	—	—	—	—	—	—	—	—
111	101·2	113·8	102·5	95·8	130	112	119	361	502	306	69·4	101·1	93·7	131	48·4	47·9	48·6	27·7	11·5
111·5	103·8	113·1	121·5	91·8	133	137	112	382	515	308	65·0	98·7	95·1	134	46·6	46·3	46·3	23·5	—
111·5	105·3	107·6	119·9	86·1	122	137	103	362	504	305	67·7	91·2	92·2	124	48·5	48·7	48·6	27·0	—
109	98·6	115·4	114·8	95·9	129	130	117	376	507	299	68·8	94·1	—	—	51·4	51·7	52·2	30·4	—
122·5	—	124·1	121·4	96·1	143	135	122	400	529	328	68·0	—	88·0	130	50·3	49·6	50·0	24·7	12·9

TABLE VIII. *Individual Measur*

L	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	O_1R	O_1L	$O_1'R$	*Lacr.* O_1R	O_2R	O_2L	G_1'	ω_1	G_2	*EH*	*fml*	*fmb*
5	24·4	—	—	—	6·3	12·3	44·0	43·8	—	—	34·0	34·5	—	—	36·7 ?	—	33·9	31·0
5	24·2	13·9	20·5	36·5	5·9	13·3	43·8	44·0	43·0	41·0	32·1	32·0	46·4	51·3	—	—	41·8	34·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35·0	28·1
9	23·4	14·9	23·5	39	5·2	11·7	44·8	41·9	40·0	39·5	32·8	31·0	47·3	54·0	—	—	—	—
8	25·6	—	—	—	3·3	12·0	39·4	40·0	—	—	30·2	31·2	44·1	49·3	33·0	11·2	37·3	28·9
4	25·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·0	31·1
3	22·8	—	—	—	—	—	40·2	42·0	—	—	37·1	36·4	39·8	44·0	—	—	38·5	28·7
4	27·0	—	—	—	4·9	12·0	42·8	43·7	—	—	30·1	29·3	51·4	54·1	43·0	11·8	35·0	31·9
	—	—	—	—	—	—	—	—	—	—	34·0	—	—	—	—	—	—	28·2
9	25·8	14·2	22·8	37	5·9	11·9	38·4	38·4	37·1	35·0	33·4	33·3	45·6	50·7	36·1 ?	—	—	—
9	25·8	13·8	20·0	34·5	4·1	9·5	40·4	41·1	38·9	—	31·5	31·1	—	—	—	—	—	—
2	27·7	—	—	—	3·4	9·6	43·1	41·9	41·4	—	32·3	32·5	51·7	56·0	42·8	15·3	36·6	27·9
9	24·5	—	—	—	6·5	10·0	38·3	39·1	—	—	27·9	28·8	44·6	48·3	41·6	10·2	29·5	28·0
1	22·5	—	—	—	5·3	12·1	40·7	42·4	—	—	33·7	33·7	47·1	52·5	43·9	15·4	35·8	—
6	24·4	11·0	22·6	34	4·0	10·0	39·9	40·6	37·0	36·9	32·1	31·1	—	48·0	36·7	13·1	32·0	27·0
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39·2	28·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·5	31·4
	—	—	—	—	—	—	41·3	41·2	—	—	30·3	30·7	—	—	—	—	31·5	32·5
4	27·2	—	—	—	3·6	11·0	43·8	43·0	39·1	39·1	30·1	30·0	50·0	53·6	40·9	12·7	33·5	32·1
0	26·8	—	—	—	3·2	12·0	40·1	41·7	—	—	—	32·9	48·9 ?	52·1 ?	—	—	37·8	28·1
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34·4	28·3
7	24·7	—	—	—	2·9	9·1	44·3	45·1	—	—	35·1	36·0	46·8	51·7	38·0	15·7	41·4	34·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39·1	—
0	28·0	—	—	—	—	—	43·5 ?	42·2	—	—	32·6 ?	—	50·0	56·2	39·5	—	41·5	30·6
5	27·5	—	—	—	3·0	12·2	40·0	40·3	—	—	32·5	32·0	42·2	47·6 ?	36·1	13·2	34·5	28·4
3	26·5	—	—	—	5·1	8·5	44·4	42·3	41·9	38·9	31·3	31·9	48·2	53·8	41·7	13·5	35·4	28·9
8	24·0	—	—	—	4·2	9·0	42·3	42·0	—	—	33·7	33·2	—	55·0	—	—	40·5	33·0
8	24·9	—	—	—	3·3	11·1	42·0	42·7	—	—	32·1	33·0	51·0	57·7	41·0	11·1	29·6	26·0
0	24·5	—	—	—	2·9	9·1	41·5	40·0	—	—	33·0	33·5	48·0	50·0	41·8	11·9	34·6	28·9
	25·2	—	—	—	—	—	39·3	39·1	—	—	31·4	32·9	47·6	55·0	36·9 ?	11·9	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37·7	27·2
0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4	26·4	13·2	23·6	35·5	3·7	9·9	40·2	40·5	36·2	—	30·3	30·0	45·9	53·2	35·8	14·0	33·8	28·4
0	27·8	12·7	25·6	38	4·0	13·1	43·7	42·9	40·1	—	36·0	35·0	52·0	57·0	—	—	33·4	28·6
4	24·5	—	—	—	—	—	41·4	40·9	—	—	34·1	34·8	—	—	—	—	33·5	27·8
6	26·1	—	—	—	5·1	12·0	42·4	42·6	—	—	32·7	34·1	—	—	44·0	9·0 ?	33·2	28·9
2	28·2	—	—	—	6·8	14·5	45·8	45·0	—	—	36·3	36·2	—	—	42·0	16·9 ?	37·8	33·5
5	27·9	—	—	—	3·2	8·5	40·7	41·3	—	—	29·7	30·1	45·4	51·0	—	—	34·7	30·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34·0	29·1
0	28·0	—	—	—	6·6	9·2	43·0	44·5	—	—	36·4	35·5	—	—	—	—	38·1	29·9
4	26·1	—	—	—	3·5	10·5	41·1	40·5	38·9	—	32·9	33·7	47·0	52·8	42·0	15·4	38·1	—
	—	—	—	—	2·2	6·9	41·4	—	—	—	34·6	—	—	—	—	—	32·3	32·1
4	25·8	14·3	24·6	38	6·7	12·8	42·5	43·1	40·7	—	34·2	33·1	49·4	52·5	40·0	—	35·4	31·1
1	26·9	8·7 ?	24·0 ?	33·5 ?	4·7	11·9	42·1	43·2	—	—	33·0	33·5	43·1	46·2	38·0	10·0	—	—
0	26·0	—	—	—	5·1	11·0	41·0	41·9	—	—	33·6	32·4	49·9	54·3	40·0	15·4	37·6	28·0
4	27·4	12·1	24·5	36·5	4·7	12·8	39·9	38·8	36·2	36·2	32·0	30·5	51·3	55·3	—	—	39·5	30·2
8	28·1	9·2	21·0	30·5	3·7	8·9	41·8	41·6	40·1	40·1	30·5	32·3	48·5 ?	52·3 ?	44·9	—	38·7	36·0
4	27·1	—	—	—	4·6 ?	10·5	42·0	41·5	—	38·0	33·5	34·3	50·7	56·2	42·8	14·0	38·2	31·2
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39·4	31·3
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39·7	34·2
5	25·9	—	—	—	1·7	7·0	40·0	41·0	—	—	32·2	32·0	46·2	51·0	41·9	13·1	34·7	28·7
9	25·6	—	—	—	2·3	10·0	44·5	41·9	41·0	40·0	32·3	32·0	50·4	54·8	40·9	—	36·2	29·1
6	26·1	—	—	—	3·9	12·5	41·1	41·0	—	—	31·5	31·0	47·4 ?	52·3 ?	38·1	—	33·8	28·9
6	23·9	—	—	—	—	—	41·4	42·6	—	—	32·0	32·7	47·4 ?	52·3 ?	41·8 ?	—	35·1	31·5
0	22·8	—	—	—	6·1	10·9	41·6	39·1	—	—	31·5	31·9	45·5	52·6	38·6	11·5	34·0	29·0
0	27·0	—	—	—	1·8	7·5	42·2	40·0	—	—	32·5	31·0	44·1	46·0	—	—	36·8	27·7
6	27·4	14·0	24·0	40	4·5	10·0	43·9	43·2	42·8	40·0	33·2	34·0	47·3	50·9	—	—	40·9	34·9
9	26·0	—	—	—	—	—	44·7	44·8	—	—	40·9	40·8	49·1	52·8	—	—	38·2	29·6
2	29·3	—	—	—	5·0	9·0	43·4	44·4	—	—	37·5	37·1	56·3	58·3	43·2	11·4	38·6	34·1
3	25·9	9·6	25·3	35	4·7	14·0	42·3	41·4	39·6	39·0	31·0	30·7	48·0	54·0	42·2	13·0	46·2	33·0
9	24·6	—	—	—	4·5	10·0	41·8	41·6	—	—	32·6	32·7	48·2	51·0	41·5	15·2	38·0	30·0
5	28·4	—	—	—	5·1	11·3	44·6	—	—	—	30·8	—	46·2 ?	52·2 ?	38·4	—	33·5	27·9
8	23·0	—	—	—	3·5	11·0	44·0	44·5	41·2	39·6	31·3	31·5	44·1	50·0	—	—	35·2	30·0
8	24·0	8·5	22·4	33·5	1·8	10·4	40·7	41·1	39·9	37·0	29·7	30·0	50·0	56·3	37·0	11·6	—	—
4	25·5	10·5	21·0	31·5	4·5	9·2	40·7	41·0	39·2	37·8	32·6	33·1	43·7	49·6	40·0 ?	—	36·1	29·9
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35·0	27·9
6	27·7	11·5	25·0	35	5·5	14·9	43·5	44·0	41·9	40·0	30·5	30·0	48·1	52·1	39·5 ?	—	35·4	29·1
3	23·5	—	—	—	2·6	9·6	42·2	42·1	—	—	29·6	30·2	—	—	—	—	37·8	29·7
6	27·0	—	—	—	1·3	8·2	39·5	39·9	—	—	31·6	30·9	45·9	49·4	—	—	—	—
2	30·4	—	—	—	3·5	9·2	42·8	—	—	—	33·1	—	49·0	53·6	40·9	12·9	35·3	30·0
0	24·7	12·9	24·0	38	5·1	13·0	40·7	41·0	39·0	37·1	31·6	31·7	43·8	49·6	36·6 ?	—	—	—

Individual Measurements of Male Kerma Skulls.

G_2	EH	fml	fmb	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{H}{L}$	100 $\frac{B}{H'}$	100 $\frac{B}{H}$	100 $\frac{(B-H')}{L}$	$Oc.\ I.$	100 $\frac{fml}{fmb}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH}, R$	100 $\frac{NB}{NH}, L$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{O_1}, L$
36·7 ?	—	33·9	31·0	74·3	73·5	74·8	101·1	99·3	0·8	61·0	94·1	75·0	47·1	46·4	47·3	—	51·2	77·3	78·8
—	—	41·8	34·7	69·9	71·9	72·9	97·2	95·8	−2·0	59·3	83·1	74·0	46·5	46·7	47·0	67·8	44·4	73·3	72·7
—	—	35·0	28·1	—	74·1	—	—	—	—	66·0	80·3	—	—	—	—	—	—	—	—
—	—	—	—	66·1	—	—	—	—	—	60·7	—	77·2	44·7	45·0	45·1	63·4	44·4	73·2	74·0
33·0	11·2	37·3	28·9	70·2	72·4	73·5	96·9	95·5	−2·2	60·0	77·5	67·3	54·5	54·4	54·7	—	27·5	76·6	78·0
—	—	36·0	31·1	69·5	—	72·9	—	95·3	—	59·9	86·4	—	49·6	48·3	49·0	—	—	—	—
—	—	38·5	28·7	71·7	75·0	75·8	95·6	94·6	−3·3	60·2	74·5	70·0	47·5	48·9	48·2	—	—	92·3	86·7
43·0	11·8	35·0	31·9	70·3	—	75·1	—	93·6	—	58·6	91·1	68·5	59·1	56·5	59·5	—	40·8	70·3	67·0
—	—	—	28·2	63·6	—	—	—	—	—	59·4	—	—	—	—	—	—	—	—	—
36·1 ?	—	—	—	70·0	—	—	—	—	—	57·5	—	70·4	52·6	52·9	53·9	62·3	49·6	86·9	86·7
—	—	—	—	70·8	—	—	—	—	—	—	—	74·4	51·0	48·9	50·7	69·0	43·1	77·9	75·7
42·8	15·3	36·6	27·9	70·8	72·7	74·1	97·4	95·6	−1·9	64·5	76·2	67·6	55·2	55·8	55·2	—	35·4	74·9	77·6
41·6	10·2	29·5	28·0	76·1	76·1	76·9	100·0	98·9	0·0	59·6	94·9	67·4	53·9	53·5	53·1	—	65·0	72·8	73·6
43·9	15·4	35·8	—	70·8	67·1	—	105·7	—	3·8	57·8	—	79·1	44·8	44·7	44·9	—	43·8	82·8	79·5
36·7	13·1	32·0	27·0	74·3	74·3	75·7	100·0	98·2	0·0	62·2	84·4	69·1	52·4	53·3	52·4	48·7	40·0	80·4	76·6
—	—	39·2	28·5	68·7	71·1	—	96·6	—	−2·4	61·7	81·0	—	—	—	—	—	—	—	—
—	—	36·5	31·4	70·1	76·3 ?	—	91·9 ?	—	−6·2 ?	64·4	86·0	—	—	—	—	—	—	—	—
—	—	31·5	32·5	73·6	74·7	75·5	98·6	97·5	−1·1	58·0	103·2	—	—	—	—	—	—	73·4	74·5
40·9	12·7	33·5	32·1	70·5	—	73·7	—	95·6	—	—	95·8	66·8	55·9	54·4	55·1	—	32·7	68·7	69·8
—	—	37·8	28·1	68·1	70·4	72·3	96·7	94·2	−2·3	57·3	74·3	—	54·7	54·1	54·7	—	28·6	—	78·9
—	—	34·4	28·3	75·8	74·1	—	102·3	—	1·7	60·3	82·3	—	—	—	—	—	—	—	—
38·0	15·7	41·4	34·5	73·2	—	71·3	—	102·6	—	—	83·3	79·0	43·9	44·1	43·6	—	31·9	79·2	79·8
—	—	39·1	—	66·6	—	—	—	—	—	58·4	—	—	—	—	—	—	—	—	—
39·5	—	41·5	30·6	72·5	77·5	78·1	93·4	92·8	−5·1	63·6	73·7	70·9	51·3	51·6	50·9	—	—	74·9 ?	—
36·1	13·2	34·5	28·4	75·3	72·8	75·0	103·4	100·4	2·5	71·8	82·3	71·3	59·8	57·9	57·9	—	24·6	81·3	79·4
41·7	13·5	35·4	28·9	72·0	—	70·4	—	102·3	—	58·8	81·6	74·3	48·9	49·1	48·8	—	60·0	71·1	75·4
—	—	40·5	33·0	67·7	73·1	74·7	92·6	90·7	−5·4	64·9	81·5	85·9	43·2	42·5	42·3	—	46·7	79·7	79·0
41·0	11·1	29·6	26·0	76·1	71·4	72·5	106·6	104·9	4·7	56·8	87·8	71·4	47·1	48·1	47·2	—	29·7	76·4	77·5
41·8	11·9	34·6	28·9	75·1	76·2	77·5	98·6	96·8	−1·1	63·6	83·5	74·0	49·4	49·8	49·0	—	31·9	79·5	82·5
36·9 ?	11·9	—	—	82·6	75·4 ?	—	109·5 ?	—	7·1 ?	62·4	—	—	—	—	—	—	—	79·9	84·1
—	—	37·7	27·2	66·5	73·5	73·5	90·5	90·5	−7·0	62·6	71·2	—	—	—	—	—	—	—	—
—	—	—	—	70·4	—	—	—	—	—	58·8	—	—	—	—	—	—	—	—	—
35·8	14·0	33·8	28·4	67·6	—	72·9	—	92·7	—	58·7	84·0	78·0	50·6	51·0	50·4	55·9	37·4	75·4	74·1
—	—	33·4	28·6	78·8	73·3	74·4	107·5	105·8	5·5	63·1	85·6	64·5	55·6	56·5	55·6	49·7	30·5	82·4	81·6
—	—	33·5	27·8	73·8	71·9	72·4	102·6	101·9	1·9	56·6	82·9	64·0	50·3	50·3	50·6	—	—	82·4	85·1
44·0	9·0 ?	33·2	28·9	64·8	71·7	73·5	90·5	88·3	−6·8	61·7	87·0	67·0	55·6	55·9	56·0	—	42·5	77·1	80·0
42·0	16·9 ?	37·8	33·5	67·9	69·7	70·7	97·5	96·1	−1·8	58·2	88·6	73·3	51·1	51·8	52·0	—	46·9	79·3	80·4
—	—	34·7	30·5	69·6	72·9	74·3	95·5	93·8	−3·3	61·2	87·9	73·4	56·4	57·4	56·4	—	37·6	72·9	72·9
—	—	34·0	29·1	70·4	—	—	—	—	—	56·4	85·6	—	—	—	—	—	—	—	—
—	—	38·1	29·9	73·2	68·0	69·4	107·6	105·5	5·2	57·1	78·5	82·7	46·7	46·9	46·7	—	71·7	84·7	79·8
42·0	15·4	38·1	—	71·2	69·4	69·9	102·6	101·8	1·8	58·1	—	78·0	51·7	50·4	49·8	—	33·3	80·0	83·2
—	—	32·3	32·1	74·3	74·9	76·2	99·3	97·5	−0·6	—	99·4	75·3	—	—	—	—	31·9	83·6	—
40·0	—	35·4	31·1	73·8	73·6	74·7	100·4	98·9	0·2	61·3	87·8	77·9	49·0	48·3	48·3	58·1	52·3	80·5	76·8
38·0	10·0	—	—	71·2	—	—	—	—	—	64·6	—	74·7	53·5	54·9	53·7	36·3 ?	39·5	78·4	77·5
40·0	15·4	37·6	28·0	68·3	—	73·0	—	93·5	—	—	74·5	77·9	49·5	51·0	50·0	—	46·4	81·9	77·1
—	—	39·5	30·2	70·7	—	75·0	—	94·3	—	—	76·5	74·5	53·3	54·6	53·3	49·4	36·7	80·2	78·4
44·9	—	38·7	36·0	70·7	69·9	70·7	101·1	100·0	0·8	59·2	93·0	75·2	53·6	51·8	53·2	43·9	41·6	73·0	77·6
42·8	14·0	38·2	31·2	65·6	—	72·2	—	91·2	—	56·5	81·7	71·2	51·0	51·6	51·7	—	43·8 ?	79·8	82·6
—	—	39·4	31·3	73·9	70·9	—	104·3	—	3·0	62·3	79·4	—	—	—	—	—	—	—	—
—	—	39·7	34·2	71·0	71·2	—	99·6	—	−0·3	57·9	86·1	—	—	—	—	—	—	—	—
41·9	13·1	34·7	28·7	68·4	—	72·4	—	94·4	—	—	82·7	72·9	53·4	52·6	52·3	—	24·3	80·5	78·0
40·9	—	36·2	29·1	75·4	—	71·9	—	104·9	—	—	80·4	71·2	48·8	48·8	49·3	—	23·0	72·6	76·4
38·1	—	33·8	28·9	72·3	71·2	71·2	101·5	101·5	1·1	61·7	85·5	76·9	52·2	52·3	52·6	—	31·2	76·6	75·6
41·8 ?	—	35·1	31·5	73·6	73·1	73·3	100·7	100·4	0·5	58·7	89·7	78·9	46·6	46·0	46·3	—	—	77·3	76·8
38·6	11·5	34·0	29·0	73·0	71·1	72·5	102·7	100·8	1·9	61·5	85·3	71·9	50·7	49·7	51·8	—	55·9	75·7	81·6
—	—	36·8	27·7	69·0 ?	67·5	68·5	102·3 ?	100·7 ?	1·5 ?	—	75·3	—	55·2	54·3	55·1	—	24·0	77·0	77·5
—	—	40·9	34·9	69·9	—	75·9	—	92·1	—	62·9	85·3	71·9	50·3	50·6	50·2	58·3	45·0	75·6	78·7
—	—	38·2	29·6	70·1	64·6	66·4	108·5	105·5	5·5	58·3	77·5	71·6	50·4	50·5	50·1	—	—	91·5	91·1
43·2	11·4	38·6	34·1	72·5	73·1	73·6	99·3	98·6	−0·5	59·2	88·3	—	53·9	53·3	52·1	—	55·6	86·4	83·6
42·2	13·0	46·2	33·0	69·5	68·1 ?	68·4	101·9 ?	101·5	1·3 ?	60·4	71·4	69·6	53·4	52·0	52·5	37·9	33·6	73·3	74·1
41·5	15·2	38·0	30·0	72·7	70·4	70·9	103·4	102·6	2·4	58·5	78·9	85·6	44·5	45·6	44·0	—	45·0	77·9	78·6
38·4	—	33·5	27·9	74·9	72·2	72·2	103·8	103·8	2·7	57·8	83·3	—	61·6	61·6	58·8	—	45·1	69·0	—
—	—	35·2	30·0	69·2	—	67·6	—	102·3	—	56·3	85·2	72·1	47·7	47·3	47·3	—	31·4	71·1	70·8
37·0	11·6	—	—	72·9	—	—	—	—	—	63·3	—	72·7	53·8	52·4	52·3	37·9	17·3	73·0	73·0
40·0 ?	—	36·1	29·9	70·7	71·2	72·3	99·2	97·8	−0·5	62·1	82·8	67·9	52·9	53·5	53·8	50·0	48·9	80·1	80·7
—	—	35·0	27·9	—	—	—	96·0	—	—	59·4	79·7	—	—	—	—	—	—	—	—
39·5 ?	—	35·4	29·1	76·1	74·2	74·2	102·7	102·7	1·9	57·4	82·2	74·1	57·2	57·8	57·0	46·0	36·9	70·1	68·2
—	—	37·3	29·7	70·5	71·6	71·3	98·5	98·9	−1·1	58·6	78·6	68·3	50·4	50·8	50·8	—	27·1	70·1	71·7
—	—	—	—	74·3	73·8	74·6	100·7	99·6	0·5	60·3	—	73·4	55·7	55·4	55·6	—	15·9	80·0	77·4
40·9	12·9	35·3	30·0	68·9	70·1	71·1	98·5	96·9	−1·1	58·6	85·0	—	59·1	58·8	58·2	—	38·0	77·3	—
36·6 ?	—	—	—	73·0	—	—	—	—	—	57·3	—	77·3	49·1	49·8	49·4	53·7	39·2	77·6	77·3

	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{O_1}, L$	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{Lacr.\ O_1}, R$	100 $\frac{G_2}{G_1'}$	100 $\frac{G_2}{G_1}$	100 $\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alv.* $P\angle$	*Pros.* $P\angle$	$\theta_1\angle$	$\theta_2\angle$	
											85°	—	—	—	Coronal and sagittal sutu
	77·3	78·8	—	—	—	—	—	—	—	—	87°	85°	31°·1	10°·5	All sutures obliterated.
	73·3	72·7	74·7	78·3	—	—	—	61°·9	76°·5	41°·6	—	—	—	—	Coronal suture closed, sa
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture partly clo
	73·2	74·0	82·0	96·7	—	—	—	—	—	—	—	—	—	—	Sagittal and lambdoid su
	76·6	78·0	—	—	74·8	66·9	33·9	62°·8	76°·9	40°·3	—	86°	—	—	Coronal and sagittal sutu
	—	—	—	—	—	—	—	65°·8	75°·9	38°·3	—	—	—	—	Coronal suture partly ob
	92·3	86·7	—	—	—	—	—	61°·5	74°·8	43°·7	85°	82°·5	33°·5	10°·2	Coronal suture obliterate
	70·3	67·0	—	—	83·7	79·5	27·4	68°·0	75°·0	37°·0	—	—	—	—	All sutures obliterated.
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture closed, sa
	86·9	86·7	90·0	95·4	83·7 ?	71·2 ?	—	—	—	—	—	87° ?	—	—	Coronal and sagittal sutu
	77·9	75·7	80·9	—	—	—	—	—	—	—	82°	81°	—	—	Coronal and sagittal sutu
	74·9	77·6	78·0	—	82·8	76·4	35·7	72°·1	69°·6	38°·3	75°	73°	32°·9	5°·4	Sagittal suture obliterate
	72·8	73·6	—	—	93·3	86·1	24·5	66°·8	77°·1	36°·1	83°	80°	30°·2	5°·9	Coronal and sagittal sutu
	82·8	79·5	—	—	93·2	83·6	35·1	61°·8	76°·2	42°·0	88°	86°	30°·2	11°·8	All sutures closing. JR.
	80·4	76·6	86·8	87·0	—	76·5	35·7	65°·8	74°·1	40°·1	84°·5	82°	29°·7	10°·4	Coronal suture beginning
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture closed, c
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sut
	73·4	74·5	—	—	—	—	—	—	—	—	—	—	—	—	All sutures obliterated.
	68·7	69·8	77·0	77·0	81·8	76·3	31·0	69°·4	70°·8	39°·8	83°	79°	27°·6	12°·2	Sagittal suture obliterate
	—	78·9	—	—	—	—	—	—	—	—	—	82°·5 ?	—	—	Sagittal suture beginnin
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures closed and a
	79·2	79·8	—	—	81·2	73·2	41·3	64°·5	73°·0	42°·5	85°	83°·5	30°·5	12°·0	Coronal and sagittal sut
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sut
	74·9 ?	—	—	—	79·0	70·3	—	64°·0	75°·0	41°·0	81°	80°·5	35°·0	6°·0	All sutures open. J=.
	81·3	79·4	—	—	85·5	75·8 ?	36·6	66°·0	73°·0	40°·8	—	84°·5	—	—	Sagittal and lambdoid s
	71·1	75·4	74·7	80·5	86·5	77·5	32·4	63°·0	71°·8	45°·2	88°	86°	29°·0	16°·2	Coronal suture beginnin
	79·7	79·0	—	—	—	—	—	59°·2	74°·9	45°·8	90° ?	87°	30°·8 ?	15°·1 ?	Sagittal suture beginnin
	76·4	77·5	—	—	80·4	71·1	27·1	71°·5	65°·5	43°·0	78°·5	76°	30°·0	13°·0	Coronal and sagittal sut
	79·5	82·5	—	—	87·1	82·1	28·5	66°·9	70°·8	42°·3	79°	76°	34°·1	8°·2	Coronal and sagittal sut
	79·9	84·1	—	—	77·5 ?	67·1 ?	32·2 ?	—	—	—	—	—	—	—	Sagittal suture partly o
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture closing.
								66°·7	72°·2	41°·0	—	—	—	—	Sagittal suture obliterat
	75·4	74·1	83·7	—	78·0	67·3	39·1	66°·2	73°·2	40°·8	86°·5	85°	27°·3	13°·3	Sagittal suture closed, c
	82·4	81·6	89·8	—	—	—	—	70°·6	72°·4	37°·0	—	—	—	—	Sagittal suture closing.
	82·4	85·1	—	—	—	—	—	60°·4	79°·8	39°·9	89° ?	—	30°·6 ?	9°·2 ?	Sagittal and coronal sut
	77·1	80·0	—	—	—	—	20·4 ?	66°·5	79°·9	33°·5	—	—	—	—	Sagittal and coronal su
	79·3	80·4	—	—	—	—	40·2 ?	64°·8	73°·5	41°·9	87°	84°	28°·2	13°·5	Sagittal suture obliterat
	72·9	72·9	—	—	—	—	—	63°·6	76°·7	39°·7	84°	81°	32°·4	7°·3	Coronal and sagittal su
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture closed, s
	84·7	79·8	—	—	—	—	—	61°·3	69°·9	48°·8	85°	84°	33°·7	15°·1	Sagittal and lambdoid
	80·0	83·2	84·6	—	89·4	79·5	30·7	65°·0	72°·9	42°·0	82°	80°	33°·0	9°·1	All sutures obliterated.
	83·6	—	—	—	—	—	—	61°·4	75°·8	42°·7	—	—	—	—	Coronal and sagittal su
	80·5	76·8	84·0	—	81·0	76·2	—	60°·0	79°·5	40°·5	90°	88°	30°·0	10°·5	All sutures closing, cor
	78·4	77·5	—	—	88·4	82·2	42·1	64°·8	74°·1	41°·1	—	83°	—	—	All sutures closed and
	81·9	77·1	—	—	80·2	73·7	38·5	66°·9	69°·0	44°·0	81°·5	79°	31°·6	12°·5	All sutures nearly oblit
	80·2	78·4	88·4	88·4	—	—	—	68°·3	70°·3	41°·4	80°	78°	31°·7	9°·7	Sagittal suture closed,
	73·0	77·6	76·1	76·1	92·6 ?	85·8 ?	—	61°·5	74°·6	43°·9	84° ?	82°·5	34°·5 ?	9°·4 ?	Coronal suture obliterat
	79·8	82·6	—	88·2	84·4	76·2	32·7	67°·2	72°·8	40°·0	85°·5	82°	27°·3	12°·7	Coronal and sagittal su
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture obliterat
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture obliterat
	80·5	78·0	—	—	90·7	82·2	31·3	69°·5	69°·5	41°·0	86°·5	83°	24°·0	17°·0	Coronal and sagittal su
	72·6	76·4	78·8	80·7	81·1	74·6	—	67°·4	71°·0	41°·7	82°	80°	30°·6	11°·0	Coronal and sagittal su
	76·6	75·6	—	—	80·4 ?	72·8 ?	—	68°·0	69°·9	42°·1	83°	80°	29°·0	13°·1	Coronal and sagittal s
	77·3	76·8	—	—	88·2 ?	79·9 ?	—	61°·0	75°·9	43°·0	86°	84°	33°·0	10°·1	J=.
	75·7	81·6	—	—	84·8	73·4	29·8	68°·0	72°·6	39°·2	83°·5	82°	28°·5	10°·9	All sutures obliterated
	77·0	77·5	—	—	—	—	—	65°·0	72°·0	43°·0	86°	84°	29°·0	14°·0	Sagittal suture obliter
	75·6	78·7	·77·6	83·0	—	—	—	61°·3	78°·1	40°·6	—	—	—	—	Coronal and sagittal s
	91·5	91·1	—	—	—	—	—	64°·3	74°·2	41°·6	88°	84°	27°·7	13°·8	J=.
	86·4	83·6	—	—	76·7	74·1	26·4	66°·5	66°·0	47°·5	79°	78°	34°·5	13°·0	Sagittal suture obliter
	73·3	74·1	78·3	79·5	87·9	78·1	30·8	63°·5	77°·5	39°·0	88°	87°·5	28°·5	10°·5	Coronal suture beginn
	77·9	78·6	—	—	86·1	81·4	36·6	63°·1	72°·8	43°·9	—	—	—	—	Coronal and sagittal s
	69·0	—	—	—	83·1 ?	73·6 ?	—	66°·0	76°·0	38°·0	82°	80°	32°·0	6°·0	Coronal and sagittal s
	71·1	70·8	75·9	79·0	—	—	—	62°·7	78°·8	38°·6	90°	89°	27°·3	11°·2	Coronal suture nearly
	73·0	73·0	77·4	80·3	74·0	65·7	31·3	—	—	—	76°	75°	—	—	Coronal suture closing
	80·1	80·7	83·2	86·2	91·5 ?	80·6 ?	—	63°·8	78°·2	38°·0	84°·5	81°·5	31°·7	6°·3	Sagittal suture obliter
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal s
	70·1	68·2	72·8	76·2	82·1 ?	75·8 ?	—	69°·8	70°·1	40°·2	79°	77°	31°·2	8°·9	Teeth worn. JR.
	70·1	71·7	—	—	—	—	—	66°·9	75°·9	37°·1	—	79°	—	—	Sagittal suture closing
	80·0	77·4	—	—	—	—	—	58°·6	81°·9	39°·5	—	86°	—	—	Sagittal suture closing
	77·3	—	—	—	83·5	76·3	31·5	65°·5	72°·9	41°·6	—	—	—	—	Sagittal suture beginn
	77·6	77·3	81·0	85·2	83·6 ?	73·8 ?	—	—	—	—	—	—	—	—	Coronal and sagittal s

$100\frac{O_2}{O_1}, R$	$100\frac{O_2}{Lacr.\,O_1}, R$	$100\frac{G_2}{G_1'}$	$100\frac{G_2}{G_1}$	$100\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alv.* $P\angle$	*Pros.* $P\angle$	$\theta_1\angle$	$\theta_2\angle$	
—	—	—	—	—	—	—	—	85°	—	—	—	Coronal and sagittal sutures closed and partly oblit
74·7	78·3	—	—	—	61°·9	76°·5	41°·6	87°	85°	31°·1	10°·5	All sutures obliterated. 4 teeth lost before death.
—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture closed, sagittal partly obliterated, la
82·0	96·7	—	—	—	—	—	—	—	—	—	—	Coronal suture partly closed. No 3rd molar on left.
—	—	74·8	66·9	33·9	62°·8	76°·9	40°·3	—	86°	—	—	Sagittal and lambdoid sutures partly obliterated, c
—	—	—	—	—	65°·8	75°·9	38°·3	—	—	—	—	Coronal and sagittal sutures partly obliterated, lam
—	—	—	—	—	61°·5	74°·8	43°·7	85°	82°·5	33°·5	10°·2	Coronal suture partly obliterated, sagittal closed. 6
—	—	83·7	79·5	27·4	68°·0	75°·0	37°·0	—	—	—	—	Coronal suture obliterated, sagittal and lambdoid p
—	—	—	—	—	—	—	—	—	—	—	—	All sutures obliterated.
90·0	95·4	83·7 ?	71·2 ?	—	—	—	—	—	87° ?	—	—	Coronal suture closed, sagittal and lambdoid nearly
80·9	—	—	—	—	—	—	—	82°	81°	—	—	Coronal and sagittal sutures nearly obliterated, lam
78·0	—	82·8	76·4	35·7	72°·1	69°·6	38°·3	75°	73°	32°·9	5°·4	Coronal and sagittal sutures closing. Teeth worn. J
—	—	93·3	86·1	24·5	66°·8	77°·1	36°·1	83°	80°	30°·2	5°·9	Sagittal suture obliterated, coronal closed, lambdoi
—	—	93·2	83·6	35·1	61°·8	76°·2	42°·0	88°	86°	30°·2	11°·8	Coronal and sagittal sutures partly closed.
86·8	87·0	—	76·5	35·7	65°·8	74°·1	40°·1	84°·5	82°	29°·7	10°·4	All sutures closing. JR.
—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture beginning to close. JR.
—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture closed, coronal beginning to close.
—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutures closed and partly oblit
77·0	77·0	81·8	76·3	31·0	69°·4	70°·8	39°·8	83°	79°	27°·6	12°·2	All sutures obliterated. Teeth worn. JR.
—	—	—	—	—	—	—	—	—	82°·5 ?	—	—	Sagittal suture obliterated. J=.
—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture beginning to close. J=.
—	—	81·2	73·2	41·3	64°·5	73°·0	42°·5	85°	83°·5	30°·5	12°·0	All sutures closed and almost obliterated. Teeth wo
—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutures partly closed.
—	—	79·0	70·3	—	64°·0	75°·0	41°·0	81°	80°·5	35°·0	6°·0	Coronal and sagittal sutures beginning to close. 1 i
—	—	85·5	75·8 ?	36·6	66°·0	73°·0	40°·8	—	84°·5	—	—	All sutures open. J=.
74·7	80·5	86·5	77·5	32·4	63°·0	71°·8	45°·2	88°	86°	29°·0	16°·2	Sagittal and lambdoid sutures nearly obliterated, c
—	—	—	—	—	59°·2	74°·9	45°·8	90° ?	87°	30°·8 ?	15°·1 ?	Coronal suture beginning to close. 1 incisor lost be
—	—	80·4	71·1	27·1	71°·5	65°·5	43°·0	78°·5	76°	30°·0	13°·0	Sagittal suture beginning to close. 2 teeth lost befo
—	—	87·1	82·1	28·5	66°·9	70°·8	42°·3	79°	76°	34°·1	8°·2	Coronal and sagittal sutures closing. Teeth worn.
—	—	77·5 ?	67·1 ?	32·2 ?	—	—	—	—	—	—	—	Coronal and sagittal sutures closing and partly obli
—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture partly obliterated, coronal closing.
—	—	—	—	—	66°·7	72°·2	41°·0	—	—	—	—	Coronal suture closing. Teeth worn.
83·7	—	78·0	67·3	39·1	66°·2	73°·2	40°·8	86°·5	85°	27°·3	13°·3	Sagittal suture obliterated. JR.
89·8	—	—	—	—	70°·6	72°·4	37°·0	—	—	—	—	Sagittal suture closed, coronal and lambdoid closin
—	—	—	—	—	60°·4	70°·8	39°·9	89° ?	—	30°·6 ?	9°·2 ?	Sagittal suture closing. JL.
—	—	—	—	20·4 ?	66°·5	79°·9	33°·5	—	—	—	—	Sagittal and coronal sutures closing. JR.
—	—	—	—	40·2 ?	64°·8	73°·5	41°·9	87°	84°	28°·2	13°·5	Sagittal and coronal sutures closing. Teeth worn.
—	—	—	—	—	63°·6	76°·7	39°·7	84°	81°	32°·4	7°·3	Sagittal suture obliterated, coronal and lambdoid c
—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutures obliterated, lambdoid
—	—	—	—	—	61°·3	69°·9	48°·8	85°	84°	33°·7	15°·1	Coronal suture closed, sagittal closing. 3 molars lo
84·6	—	89·4	79·5	36·7	65°·0	72°·9	42°·0	82°	80°	33°·0	9°·1	Sagittal and lambdoid sutures closing. JR.
—	—	—	—	—	61°·4	75°·8	42°·7	—	—	—	—	All sutures obliterated. J=.
84·0	—	81·0	76·2	—	60°·0	79°·5	40°·5	90°	88°	30°·0	10°·5	Coronal and sagittal sutures closed and partly obli
—	—	88·4	82·2	42·1	64°·8	74°·1	41°·1	—	83°	—	—	All sutures closing, coronal partly obliterated. Teet
—	—	80·2	73·7	38·5	66°·9	69°·0	44°·0	81°·5	79°	31°·6	12°·5	All sutures closed and partly obliterated. Teeth wo
88·4	88·4	—	—	—	68°·3	70°·3	41°·4	80°	78°	31°·7	9°·7	All sutures nearly obliterated. 7 teeth lost before
76·1	76·1	92·6 ?	85·8 ?	—	61°·5	74°·6	43°·9	84° ?	82°·5	34°·5 ?	9°·4 ?	Sagittal suture closed, coronal and lambdoid closin
—	88·2	84·4	76·2	32·7	67°·2	72°·8	40°·0	85°·5	82°	27°·3	12°·7	Coronal suture obliterated, sagittal and lambdoid
—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutures closing. JR.
—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture obliterated, coronal and lambdoid
—	—	90·7	82·2	31·3	69°·5	69°·5	41°·0	86°·5	83°	24°·0	17°·0	Coronal suture obliterated, sagittal closed. Teeth w
78·8	80·7	81·1	74·6	—	67°·4	71°·0	41°·7	82°	80°	30°·6	11°·0	Coronal and sagittal sutures partly obliterated, lan
—	—	80·4 ?	72·8 ?	—	68°·0	69°·9	42°·1	83°	80°	29°·0	13°·1	Coronal and sagittal sutures closing. Teeth worn.
—	—	88·2 ?	79·9 ?	—	61°·0	75°·9	43°·0	86°	84°	33°·0	10°·1	Coronal and sagittal sutures closing. Teeth worn a
—	—	84·8	73·4	29·8	68°·0	72°·6	39°·2	83°·5	82°	28°·5	10°·9	J=.
—	—	—	—	—	65°·0	72°·0	43°·0	86°	84°	29°·0	14°·0	All sutures obliterated. Teeth worn. JR.
77·6	83·0	—	—	—	61°·3	78°·1	40°·6	—	—	—	—	Sagittal suture obliterated, coronal and lambdoid
—	—	—	—	—	64°·3	74°·2	41°·6	88°	84°	27°·7	13°·8	Coronal and sagittal sutures closed and partly obli
—	—	76·7	74·1	26·4	66°·5	66°·0	47°·5	79°	78°	34°·5	13°·0	J=.
78·3	79·5	87·9	78·1	30·8	63°·5	77°·5	39°·0	88°	87°·5	28°·5	10°·5	Sagittal suture obliterated, coronal and lambdoid
—	—	86·1	81·4	36·6	63°·1	72°·8	43°·9	—	—	—	—	Coronal suture beginning to close. JR.
—	—	83·1 ?	73·6 ?	—	66°·0	76°·0	38°·0	82°	80°	32°·0	6°·0	Coronal and sagittal sutures beginning to close. Te
75·9	79·0	—	—	—	62°·7	78°·8	38°·6	90°	89°	27°·3	11°·2	Coronal suture nearly obliterated, sagittal closed,
77·4	80·3	74·0	65·7	31·3	—	—	—	76°	75°	—	—	Coronal suture closing.
83·2	86·2	91·5 ?	80·6 ?	—	63°·8	78°·2	38°·0	84°·5	81°·5	31°·7	6°·3	Sagittal suture obliterated, lambdoid and coronal
—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutures closing. JR.
72·8	76·2	82·1 ?	75·8 ?	—	69°·8	70°·1	40°·2	79°	77°	31°·2	8°·9	Teeth worn. JR.
—	—	—	—	—	66°·9	75°·9	37°·1	—	79°	—	—	Sagittal suture closing. Teeth worn. JR.
—	—	—	—	—	58°·6	81°·9	39°·5	—	86°	—	—	Sagittal suture closing. Teeth markedly worn and
—	—	83·5	76·3	31·5	65°·5	72°·9	41°·6	—	—	—	—	Sagittal suture beginning to close. Teeth worn. JI
81·0	85·2	83·6 ?	73·8 ?	—	—	—	—	—	—	—	—	Coronal and sagittal sutures beginning to close.

Remarks
l and partly obliterated. JR.
st before death. JR.
tly obliterated, lambdoid beginning to close. JR.
3rd molar on left.
tly obliterated, coronal closing. JL.
y obliterated, lambdoid beginning to close. JL.
sagittal closed. 6 molars lost before death. JR.
l and lambdoid partly obliterated. JR.
l lambdoid nearly obliterated. 1 molar lost before death.
y obliterated, lambdoid nearly closed. 6 teeth lost before death.
g. Teeth worn. JR.
l closed, lambdoid closing. Teeth worn. JR.
y closed.
JR.
ginning to close. JL.
d and partly obliterated. JR.
n. JR.
J=.
terated. Teeth worn. JL.
y closed.
ning to close. 1 molar lost before death. Teeth worn. JR.
rly obliterated, coronal closed. Teeth worn. J=.
1 incisor lost before death. Teeth worn. JR.
2 teeth lost before death. JL.
g. Teeth worn. JL.
g and partly obliterated. 4 or 5 teeth lost before death.
coronal closing. 1 tooth lost before death. JR.
n.
l lambdoid closing. 6 teeth lost before death. JR.
g. JR.
g. Teeth worn. JR.
al and lambdoid closing. Several teeth lost before death. JR.
erated, lambdoid closed. J=.
sing. 3 molars lost before death. Teeth worn. JR.
sing. JR.
d and partly obliterated, lambdoid closing. Teeth worn. JL.
obliterated. Teeth worn.
terated. Teeth worn. JR.
teeth lost before death. JR.
d lambdoid closing. Teeth worn. J=.
al and lambdoid nearly obliterated. Teeth markedly worn and 1 or more lost before death. JL.
ng. JR.
al and lambdoid closing. J=.
al closed. Teeth worn. JR.
ly obliterated, lambdoid closing. Teeth worn. JL.
ng. Teeth worn. J=.
ng. Teeth worn and 2 lost before death. J=.
rn. JR.
al and lambdoid closed. Teeth worn and 4 lost before death. JR.
ed and partly obliterated, lambdoid closing. Teeth markedly worn and 1 or more lost before death. JR.
al and lambdoid closed. Teeth worn and 1 lost before death. JR.
. JR.
nning to close. Teeth worn. JR.
, sagittal closed, lambdoid closing. 5 teeth lost before death. JR.
doid and coronal closing. Teeth worn. JL.
ng. JR.
rn. JR.
rkedly worn and 1 molar lost before death. JR.
e. Teeth worn. JR.
nning to close.

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Serial No.	Grave Mark	Probable Dynasty	*C*	*B'*	*B''*	*B*	*Bias-terionic B*	*F*	*L*	*H'*	*H*	*OH*
74	1801 D	XIII	1406·5	89·3	121	139·5	—	176·5	178	—	133	108
75	437 C	XII	—	94·7	—	137·5	107·7 ?	178	180	127·5	129	107
76	430 X	„	1329	91·6	—	133	111·8	179·5	184	129	132·5	108·5
77	III 301 (XB)	„	—	99·1	115 ?	135	105·0 ?	184	187	—	—	105·5
78	2361 A	XIII	1436·5	85·0	113	137	—	181	186	—	137	109
80	1422	„	1572·5	91·5	114	138	106·6	189	189·5	132·5	134	110·5
81	2330 G	„	—	91·3	114	140·5	110·1 ?	178·5	182	138	140·5	111
82	2000 C	„	1440	96·8	114 ?	134	104·9	185	189	136	137	112
83	1805 N	„	—	88·1	104	124	102·6	181	185	134	135	110
84	333 A	XII	1340	92·3	—	129·5	105·0	182	189	130	134	107·5
85	1619 B (i)	XIII	1533·5	89·0	114·5	142	105·7	186	190·5	138	139	115
86	1065/G	„	—	89·4	—	129	110·0	183·5	187	—	135	106
87	1086/D	„	—	99·3	—	140·5	—	192	195	141	144	115
88	1075/D	„	1629·5	96·6	—	140	111·2 ?	192	193	—	144	112·5
89	1067/C	„	—	88·5	—	130 ?	—	167	—	138 ?	—	111·5
90	1619 (*d*)	„	—	93·6	—	—	103·7	187·5	190·5	—	141	111·5
91	1601	„	—	90·2	111·5	128	100·9	174	179	136·5	140	109
92	1088/A	„	1528	88·7	113·5	139·5	104·2	185	187	142	146	120
93	425 (ii)	XII	—	90·8	—	127	—	190	192·5	—	—	105·5
94	1604 K	XIII	—	98·2	116	136	111·8	192·5	196	135	138	110
95	424	XII	1505·5	94·7	—	143·5	113·4	190	193	136	138·5	112
96	420 A"	„	—	88·4	—	129·5	108·6 ?	186·5	187·5	—	—	—
97	307 A	„	1180·5	94·3	111·5	132	97·2 ?	172	173	126·5	126·5	105
98	XB/QC	XIII	—	96·4	—	133	98·7	181	184	—	—	—
99	XB/OC (A)	„	1426	86·0	—	138·5	98·2	175·5	178	131 ?	133	113·5
100	XB/E	„	—	93·1	113	138	99·8	179	178	125·5	127	106·5
101	XB/EA	„	1449·5	93·1	118	132	104·6	181·5	182	135	137	112·5
102	XB/VA	„	1251	95·5	—	125	101·4	186	187	—	—	110
103	XB/DC	„	1372	88·4	—	139	106·4	185	185	134·5	135	114·5
104	XB/XC (A)	„	1242·5	93·2	120·5	134	105·8 ?	177	175	127·5	131	109
105	8 B	XII	1369	93·7	113	140	103·3 ?	173	174	136	137	108
106	326 B	„	—	88·8	—	—	—	173·5	175·5	—	129·5	108·5
107	1088/B	XIII	—	85·1	—	—	—	179·5	182	135 ?	—	—
108	333 B	XII	—	—	—	132	104·3	177	179	122 ?	—	—
109	1048/B	XIII	1283·5	92·8	115·5	136	110·2	182	184	129	132	109·5
110	1802 B	„	—	81·8	104	132	—	177·5	180	—	—	—
111	IV B/WA	XII	—	—	—	131·5 ?	103·0 ?	—	—	—	—	—
112	IV B/DA	„	—	88·5	—	126·5	—	183·5	185·5	138 ?	139·5	111·5
113	309	„	—	94·8	111·5	135	—	173	173	—	—	107
114	308 B	„	—	91·2	—	132	106·7	192	193	—	133·5	108·5
115	421 B	„	1268	94·2	111·5	131	96·2	173	173	134·5	135	110
116	417	„	—	93·0	105	125·5	104·0	176	176·5	126·5	—	105·5
117	III 301 (XB)	„	1340	93·2	—	129	96·8	183·5	185	139	138	109
118	IV B/HA	„	—	93·7	118	131	95·3 ?	185	184·5	—	—	112·5
119	414 A	„	1350·5	93·9	—	127	102·7	188	187	—	130	103
120	1029	XIII	1480	91·0	115	134	111·3	190	190·5	132·5	133	107·5
121	1805 C	„	—	91·1	—	—	—	178·5	182	—	—	110·5
122	1805 A	„	—	94·6	113	136	107·9	177·5	177	131	132	105
123	445 (i)	XII	—	95·6	—	132	—	187	188·5	126	127	106·5
124	430	„	—	84·8	—	127	98·7 ?	175	178	127	132	106·5
126	2329 D	XIII	—	91·0	—	137 ?	—	176	179	—	—	—
127	437 X/B	XII	1388	92·5	—	136	101·8	183·5	184·5	135·5 ?	137	113
128	430	„	1381	91·6	—	142·5	105·7	181·5	182	—	127	108
130	425 C	„	1208	87·2	—	125	98·6	178	179	129	131	105·5
131	1090	XIII	—	91·8	111	138	108·1	180·5	183	134	136	112·5
132	444 H	XII	1278·5	94·6	107·5	126	100·8	183·5	184	122·5	124·5	108
133	337 A	„	—	100·2	—	139·5	111·0	187·5	188	—	—	113
134	1619 C	XIII	—	97·1	117	135	104·2	185	186·5	—	—	109·5
135	1625 D	„	—	94·0	—	128·5	—	180	182	—	—	—
136	326 A	XII	—	95·2	113·5	132	102·8	185·5	187·5	130·5	—	109
138	1600 C	XIII	1364·5	95·4	122	139	102·6	182	182	141	141·5	115
139	2355	„	1405	93·1	—	140	107·5	178·5	182	137	—	115·5
140	1600 C	„	1469	93·3	112	139·5	—	183	181	133	137	114
141	B 21	XII	—	92·0	—	134	—	182·5	184	—	—	101
142	1802 C	XIII	1392·5	92·4	115	136	103·1	177	180	131·5	133	106·5
143	1600 C/T	„	—	89·0	—	130·5	—	171	175	—	—	102
144	3901 K	„	—	90·4	114	133·5	99·4 ?	184	184	130	129	108·5
145	1619 A	„	1266·5	91·0	111	130·5	95·4 ?	171	172	130	130	104·5
146	1805 A	„	1503·5	89·0	—	137·5	103·4 ?	181	182	137	139	112
147	2312 G	„	1329·5	96·2	105·5	131·5	106·2	178	180	136	140	112·5

H	*OH*	*LB*	S_1'	S_2'	S_3'	S_1	S_2	S_3	*S*	*U*	*Q'*	*G'H*	*GL*	*GB*	*J*	*NH'*	*NH, R*	*NH, L*	*NI*
133	108	98·7	111·4	111·2	88·7	130	127	106	363	505	309	—	—	93·0	124	46·1	46·1	45·8	27·
129	107	94·4	106·4	115·0	96·4	119	126	120	365	503	299	—	—	—	—	—	—	—	—
132·5	108·5	100·5	106·7	110·2	104·2	119	121	124	364	506	299	65·3	96·0	99·3	126	52·2	53·3	53·2	29·
—	105·5	—	119·1	106·4	94·0	137	116	116	369	517	299	69·7	—	101·0	126·5	50·4	50·1	50·7	25·
137	109	99·0	114·2	115·7	95·7	131	133	116	380	504	303	—	—	89·6	119	46·1	47·0	46·8	22·
134	110·5	99·8	122·6	119·2	95·2	139	135	115	389	528	302	71·6	94·6	95·8	129·5	53·1	53·2	53·0	25·
140·5	111	100·6	113·2	112·1	98·6	125	125	114	364	512	305	76·7	101·7	99·0	130	54·4	53·9	53·9	25·
137	112	106·8	111·9	116·0	96·0	125	127	117	369	517	308	66·8	98·4	103·2	—	51·5	51·2	51·9	27·
135	110	106·1	106·1	112·7	91·5	119	127	107	353	495	291	59·6	100·4	89·7	122	45·6	46·2	47·0	26·
134	107·5	106·9	110·8	114·5	91·2	125	131	107	363	507	295	66·1	98·8	100·5	132	49·0	49·5	49·2	26·
139	115	100·8	117·3	122·0	105·6	133	136	125	394	524	318	74·3	99·5	101·4	133	51·4	52·1	52·3	28·
135	106	105·5	112·3	118·4	95·7	127	131	113	371	512	293	72·1	100·5	92·0	—	53·7	53·4	54·5	25·
144	115	105·4	118·1	122·6	99·8	135	138	121	394	538	318	73·0	92·6	99·0	133	53·2	52·1	53·1	24·
144	112·5	105·4	114·7	128·0	105·2	132	144	125	401	531	315	71·5	98·8	102·6	136	49·2	48·4	48·9	25·
—	111·5	—	—	119·0	90·0	—	138	104	—	497?	304	—	—	—	—	—	—	—	—
141	111·5	103·8	—	—	—	—	—	—	380	521?	—	72·5	95·5	92·1	127	50·4	51·0	50·7	25·
140	109	102·7	110·8	116·3	91·8	122	134	110	366	489	297	65·8	92·2	103·4	—	50·4	50·6	50·8	28·
146	120	103·5	117·2	120·1	96·7	131	138	117	386	508	324	66·8	86·6	88·9	128·5	51·0	51·4	50·8	23·
—	105·5	102·3	107·5	116·4	100·1	121	131	123	375	517?	286	69·0	—	94·7	—	—	51·5	51·9	26·
138	110	112·6	111·6	111·7	107·2	126	119	135	380	535	309	75·1	106·0	102·2	137	53·4	51·5	53·6	26·
138·5	112	100·5	117·6	120·9	103·2	134	135	125	394	534	314	68·4	94·2	97·4	137	49·4	49·6	49·2	28·
—	—	—	114·6	110·8	107·3	129	120	130	379	509?	—	67·7	—	95·3	—	—	47·2	48·1	25·
126·5	105	94·5	107·7	114·0	88·4	123	131	108	362	482	296	62·2	89·4	93·3	121	46·3	46·9	46·0	23·
—	—	—	110·1	107·6	108·2	123	119	131	373	506	—	—	—	—	—	—	—	—	—
133	113·5	93·0	110·9	116·0	98·1	125	130	115	370	498	319	69·7	91·0	95·6	126·5	50·5	50·3	50·4	25·
127	106·5	94·8	108·4	105·6	101·6	125	117	126	368	500	302	74·4	94·1	—	—	52·3	52·8	52·1	28·
137	112·5	103·2	119·1	116·0	89·5	139	131	105	375	504	310	66·8	102·8	88·2	124	49·5	49·8	49·4	26·
—	110	99·5	98·3	124·2	108·0	111	140	131	382	511	296	—	—	—	—	—	—	—	—
135	114·5	100·5	110·1	130·7	92·4	125	146?	105?	376	507	313	68·5	92·8	90·7	122?	50·0	50·1	50·1	27·
131	109	99·1	109·3	111·3	84·5	128	127	100	355	502	305	65·5	—	87·6	122	47·6	47·2	47·4	26·
137	108	102·0	107·8	112·3	90·8	123	128	107	358	495	308	67·0	92·8	93·2	—	50·3	50·0	49·8	24·
129·5	108·5	99·0	109·6	112·3	90·3	126	123	108	357	—	303	68·4	89·0	91·4	118	—	48·5	49·0	24·
—	—	104·6	111·6	113·7	91·7	134	122	106	362	—	—	72·2	97·5	80·9	—	—	49·9	50·0	23·
—	—	97·8	107·6	109·7	93·5	122	125	113	360	—	—	—	—	—	—	—	—	—	—
132	109·5	96·6	113·0	113·6	93·9	128	128	114	370	505	303	65·0	95·7	97·4	120·5	43·6	45·0	44·0	28·
—	—	—	105·5	105·3	101·7	118	117	121	356	486?	—	67·2	—	96·3	—	47·0	47·8	46·4	24·
—	—	—	—	119·9	—	—	135	—	—	—	—	—	—	—	—	—	—	—	—
139·5	111·5	106·7	113·9	110·8	95·6	128	124	112	364	502	301	72·8	97·2	93·3	—	—	51·2	51·3	24·
—	107	—	97·6	110·7	—	114	123	—	—	487	304	60·4	—	88·5	—	43·2	43·4	43·7	22·
133·5	108·5	104·6	115·1	121·3	92·7	132	136	115	383	525	302	67·1	95·5	97·7	—	50·3	50·4	51·1	23·
135	110	97·6	109·2	114·9	88·7	127	131	109	367	485	306	62·7	92·0	95·8	122	44·4	45·8	44·6	27·
—	105·5	91·8	103·3	115·1	—	116	127	—	—	485	293	74·5	88·5	99·2	122	50·6	50·5	50·8	26·
138	109	106·5	108·9	117·5	96·7	126	128	113	367	507	297	70·9	99·1	97·0	122·5	49·4	51·3	49·5	24·
—	112·5	—	109·5	106·7	95·3	126	119	113	358	499	316	66·5	—	91·2	—	48·4	48·5	47·8	25·
130	103	104·4	105·6	120·1	94·3	125	132	116	373	509	289	66·0	97·7	94·7	127·5	51·2	50·7	51·2	27·
133	107·5	101·3	116·0	118·2	96·3	132	129	115	376	520	294	—	—	97·2	127·5	—	—	—	28·
—	110·5	—	115·0	101·4	101·1	131	110	123	364	—	—	66·4	—	100·4	—	—	—	—	25·
132	105	100·1	101·8	113·0	96·4	116	129	115	360	501	297	73·6	92·8	90·1	125	52·1	51·7	51·0	23·
127	106·5	99·7	119·0	113·1	92·0	139	125	113	377	519	295	81·4	98·2	95·6	—	54·9	53·5	54·3	27·
132	106·5	102·1	100·2	107·2	94·7	112	120	116	348	485	293	70·3	96·7	89·6	122·5	51·1	50·0	50·6	27·
—	—	—	105·8	106·9	—	121	124	—	—	—	—	—	—	—	—	—	—	—	—
? 137	113	99·1	118·0	118·1	100·8	133	130	121	384	509	311	66·0	99·6	96·1	122	47·8	47·5	47·4	25·
127	108	95·9	111·3	108·5	103·4	133	117	124	374	509	305	65·7	85·6	91·7	123	47·6	48·4	47·8	25·
131	105·5	94·1	103·8	112·7	103·8	117	125	127	369	490	288	62·6	89·9	92·4	122	47·5	47·1	47·4	27·
136	112·5	103·3	108·0	115·2	96·8	121	130	120	371	511	308	—	102·5	—	—	—	—	—	—
5 124·5	108	102·9	110·4	112·6	90·2	126	128	109	363	501	295	—	—	—	—	—	—	—	—
—	113	110·5	113·4	120·8	98·7	130	138	112	380	528	314	71·6	101·1	99·7	—	—	53·0	52·4	26·
—	109·5	—	110·8	117·3	94·9	126	130	112	368	521	308	77·8	—	96·5	127	51·2	50·2	50·6	25·
—	—	—	—	—	—	—	—	—	362	504	—	—	—	—	—	—	—	—	—
5 —	109	95·0	110·8	124·0	100·6	126	140	119	385	514	300	—	—	—	—	—	—	—	—
141·5	115	100·9	118·6	117·4	96·7	139	131	111	381	506	314	63·2	96·6	98·3	125	47·1	46·4	46·6	23·
—	115·5	99·6	114·0	119·9	101·2	129	139	113	381	509	320	—	—	—	—	—	—	—	—
137	114	99·5	108·8	117·3	95·3	130	138	115	383	513	314	67·5	96·5	92·2	124·5	48·7	48·4	48·0	25·
—	101	95·4	—	—	—	—	—	—	363	510	289	—	—	—	—	—	—	—	—
5 133	106·5	100·3	107·6	112·0	94·1	121	127	110	358	504	299	70·0	94·4	96·9	133·5	49·6	48·4	48·2	25·
—	102	—	105·4	98·1	—	119	106	—	—	486	283	72·3	—	100·4	131	53·3	53·3	53·2	29·
129	108·5	97·6	115·7	112·0	89·5	134	127	110	371	504	304	65·3	92·3	94·4	—	45·7	45·8	45·9	23·
130	104·5	99·7	104·6	107·2	91·2	121	122	106	349	480	294	62·4	94·6	94·5	123·5	45·7	44·7	45·0	23·
139	112	102·0	106·8	114·8	103·3	121	127	120	368	504	311	71·2	94·6	99·5	—	47·7	48·3	47·8	25·
140	112·5	104·6	109·1	114·3	94·0	123	130	110	363	494	306	72·9	106·4	91·0	112·5	50·4	49·3	50·8	24·

TABLE VIII. *Individual Measurements o*

, *R*	*NH, L*	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	O_1R	O_1L	$O_1'R$	*Lacr.* O_1R	O_2R	O_2L	G_1'	G_1	G_2	*EH*	*fml*
·1	45·8	27·7	—	—	—	—	—	40·6	41·0	—	—	34·0	35·0	46·7	52·3	—	—	37·2
—	—	—	—	—	—	—	—	44·9	—	—	—	35·9	—	—	—	—	—	35·4
·3	53·2	29·5	10·7	22·9	33	7·3	14·0	41·0	42·0	40·0	—	33·2	34·0	—	—	—	—	38·1
·1	50·7	25·3	13·0	26·1	37	4·3	11·9	41·5	42·4	38·8	—	30·4	31·3	51·0 ?	—	39·5	10·4	—
·0	46·8	22·1	—	—	—	1·7	7·0	37·6	38·0	—	—	31·7	30·0	—	—	—	—	34·3
·2	53·0	25·0	—	—	—	—	—	42·9	42·7	—	—	34·2	34·0	44·1	47·8	41·6	14·2	35·3
·9	53·9	25·8	—	—	—	4·6	10·0	43·0	42·9	—	—	34·6	35·5	48·9	56·8	—	—	39·8
·2	51·9	27·1	—	—	—	3·9	12·3	43·2	43·0	—	—	32·3	32·9	42·2	49·5	40·4 ?	—	35·7
·2	47·0	26·4	—	—	—	3·5	11·6	40·7	39·9	—	—	30·6	30·8	—	—	—	—	42·9
·5	49·2	26·1	—	—	—	3·3	9·5	42·0	41·0	—	—	32·5	34·1	43·3	49·5	41·0	10·5	37·5
·1	52·3	28·3	—	—	—	2·9	10·0	39·8	40·0	—	—	33·1	34·6	50·0	55·3	40·0	11·6	35·2
·4	54·5	25·0	—	—	—	3·5 ?	8·1 ?	43·7	45·4	—	—	35·2	34·6	—	—	—	—	38·5
·1	53·1	24·8	—	—	—	4·8	11·2	43·4	43·5	—	—	34·3	34·9	47·9	51·8	39·6	14·1	39·5
·4	48·9	25·6	10·1	21·0	32	4·7	9·5	43·5	44·0	41·6	39·2	34·0	34·0	50·5	55·9	39·5	—	37·8
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·9 ?
·0	50·7	25·8	14·3	20·3	37	5·6	13·2	40·9	42·0	40·0	37·2	30·7	29·0	45·7	50·8	38·0	17·1	32·1
·6	50·8	28·9	—	—	—	3·3	10·8	41·8	41·0	40·0	38·0	30·3	30·1	42·5	47·3	46·7	12·8	35·3
·4	50·8	23·4	—	—	—	5·7	8·8	41·9	41·7	39·6	—	34·4	33·2	—	—	—	—	36·3
·5	51·9	26·2	—	—	—	—	—	42·0	—	—	—	32·1	—	—	—	39·2	—	37·6
·5	53·6	26·8	12·1	24·1	34	4·1	12·9	44·5	43·3	42·1	41·2	32·9	33·8	50·9	57·0	45·3	19·9 ?	36·2
·6	49·2	28·0	10·1	21·2	31	3·5	9·7	41·5	41·5	41·3	—	32·1	31·0	45·9	50·0	46·5	13·8	40·0
·2	48·1	25·1	—	—	—	—	—	41·5	—	—	—	31·0	—	—	—	—	—	—
·9	46·0	23·0	—	—	—	5·4	12·0	40·0	38·7	—	—	31·2	32·9	44·0	50·6	34·0 ?	—	29·7
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
·3	50·4	25·0	—	—	—	3·9	11·2	39·8	39·5	37·8	36·7 ?	32·9	33·5	48·0	51·3	36·9	13·1	36·8
·8	52·1	28·5	—	—	—	—	—	41·8	—	—	—	35·1	—	—	—	35·0	—	34·3
·8	49·4	26·3	—	—	—	2·3	11·0	43·1	41·2	—	—	31·1	32·1	53·2	58·5	38·9	11·1	37·8
-	—	—	—	—	—	—	—	43·5	—	—	—	—	—	48·1	—	—	—	34·7
·1	50·1	27·1	13·1 ?	23·2 ?	37 ?	—	—	39·4	38·6	36·0 ?	—	29·1	30·0	43·0	48·1	36·5	12·0	32·3
·2	47·4	26·7	8·1	21·8	29·5	3·7	12·0	39·4	40·2	37·6	—	33·2	33·0	44·6	47·2	36·0	10·5	37·5
·0	49·8	24·0	—	—	—	2·7	10·0	40·4	—	—	—	30·3	—	44·0	49·0	39·5	11·2	34·3
·5	49·0	24·1	—	18·6	—	—	—	39·0	38·2	37·7	—	33·3	31·9	—	—	38·3	11·3	36·6
·9	50·0	23·6	—	—	—	—	—	—	—	—	—	—	—	—	—	32·2 ?	—	38·4
-	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·1
·0	44·0	28·3	—	—	—	3·9	10·1	42·0	42·4	38·0	—	28·5	29·0	46·4	52·1	40·0	14·2	34·0
·8	46·4	24·0	—	—	—	—	—	39·3	—	—	—	31·7	—	—	—	40·0	11·3	—
-	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
·2	51·3	24·3	—	—	—	—	—	—	40·3 ?	—	—	—	33·9 ?	—	—	36·0	—	—
·4	43·7	22·2	7·8	24·3	30	3·7	11·9	39·0	40·4	38·0	37·1	32·1	32·1	43·7	47·0	33·6	13·6	—
·4	51·1	23·8	—	—	—	3·8	11·5	39·5	39·8	—	—	32·5	33·4	46·5 ?	50·2 ?	—	—	38·4
·8	44·6	27·5	10·7	25·9	36	6·5	15·0	37·2	38·2	35·4	34·9	30·6	30·8	44·2	48·3	38·1	14·8 ?	33·4
·5	50·8	26·0	—	—	—	—	—	39·2	39·0	—	—	35·7	35·1	46·1	49·7	39·4 ?	—	—
·3	49·5	24·5	—	—	—	5·6	13·0	42·8	43·2	—	—	35·3	35·4	48·7	56·3	35·7 ?	—	39·1
·5	47·8	25·4	—	—	—	3·7	10·1	41·9	41·0	—	—	28·5	29·6	—	—	40·9	—	—
·7	51·2	27·0	9·1 ?	24·0 ?	29·5 ?	3·4	11·6	42·5	41·2	40·6	—	31·1	31·0	43·0 ?	48·6 ?	—	—	34·2
-	—	28·6	10·3	21·8	32	2·0	8·2	43·2	44·6	42·0	40·0	31·0	31·4	—	—	41·5	12·5	42·8
-	—	25·4	—	—	—	—	—	41·3	—	—	—	34·4	—	—	—	—	—	—
·7	51·0	23·0	—	—	—	4·1	9·9	41·4	41·2	—	—	34·4	35·0	51·1	55·0	38·2	12·7	34·5
·5	54·3	27·2	—	—	—	3·4	15·5	41·0	40·7	—	—	35·8	37·0	49·5	52·0 ?	36·0	15·2 ?	35·1
·0	50·6	27·2	—	—	—	6·9	14·0	40·0	40·5	—	—	31·8	32·0	50·9	55·0	39·1	15·2	33·4
-	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
·5	47·4	25·5	—	—	—	4·3	15·5	38·5	38·5	36·9	35·9	31·4	32·0	52·2	56·6	—	—	34·9
·4	47·8	25·2	13·9	18·9	29	2·3	11·0	40·2	40·7	39·0	—	33·4	33·9	44·0	47·2	35·0	—	35·7
·1	47·4	27·8	—	—	—	3·5	9·8	41·5	40·9	—	—	33·8	34·2	44·7 ?	—	—	—	33·8
-	—	—	—	—	—	—	—	41·8	—	—	—	32·8	—	—	—	40·5	—	35·1
-	—	—	—	—	—	—	—	—	43·0	—	—	—	34·0	—	—	—	—	35·5
·0	52·4	26·2	—	—	—	—	—	42·1	—	—	—	37·0	—	—	—	41·5	12·1 ?	—
·2	50·6	25·2	—	—	—	2·3	5·9	44·2	43·4	—	—	34·7	34·8	49·6	56·5	40·1	18·3	—
-	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
-	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	45·0
·4	46·6	23·8	—	—	—	1·5	10·0	42·0	42·7	38·6	37·9	32·2	31·6	45·4	47·0	37·6	11·1	34·8
·4	48·0	25·0	—	—	—	2·9	8·4	41·2	41·2	—	—	34·1	33·3	48·2	55·0	41·5	13·9	31·5
-	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
·4	48·2	25·3	13·9	22·6	38·5	5·2	11·4	39·6	39·0	37·0	37·0	34·1	34·5	47·7	51·2	36·0	13·0	36·9
·3	53·2	29·6	10·0	19·1	33	2·3	8·5	45·4	45·2	42·4	—	—	34·3	—	—	42·4	—	—
·8	45·9	23·0	—	—	—	3·8	14·3	37·8	38·0	36·8	—	32·2	31·2	44·4	50·4	37·8 ?	—	38·8
·7	45·0	23·0	—	—	—	3·4	9·2	41·1	41·0	—	—	31·1	31·2	44·2 ?	47·3 ?	—	—	—
·3	47·8	25·0	—	—	—	3·0	9·0	41·9	41·2	—	—	35·2	35·0	46·7	51·0	—	—	39·5
·5	50·8	24·7	—	—	—	2·5	8·4	39·8	41·6	—	—	34·0	34·0	53·2	58·5	38·0 ?	—	—

Measurements of Male Kerma Skulls (continued).

G_3	EH	fml	fmb	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{H}{L}$	100 $\frac{B}{H'}$	100 $\frac{B}{H}$	100 $\frac{(B-H')}{L}$	*Oc. I.*	100 $\frac{fmb}{fml}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{\overline{NH}, R}$	100 $\frac{NB}{\overline{NH}, L}$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1, R}$	100 $\frac{O_2}{O_1}$
—	—	37·2	32·1	78·4	—	74·7	—	104·9	—	60·4	86·3	—	60·1	60·1	60·5	—	—	83·7	85
—	—	35·4	27·0	76·4	70·8	71·7	107·8	106·6	5·5	57·3	76·3	—	—	—	—	—	—	80·0	—
—	—	38·1	33·1	72·3	70·1	71·7	103·1	100·4	2·2	60·8	86·9	65·8	56·5	55·3	55·5	46·7	52·1	80·9	80
39·5	10·4	—	—	72·2	—	—	—	—	—	57·8	—	69·0	50·2	50·5	49·9	49·8	36·1	73·3	73
—	—	34·3	27·2	73·7	—	73·7	—	100·0	—	59·2	79·3	—	47·9	47·0	47·2	—	24·3	84·3	78
41·6	14·2	35·3	31·8	72·8	69·9	70·7	104·1	102·9	2·9	59·4	90·0	74·7	47·1	47·0	47·2	—	—	79·7	79
—	—	39·8	32·0	77·2	75·8	77·2	101·8	100·0	1·4	64·2	80·4	77·5	47·4	47·9	47·9	—	46·0	80·5	82
40·4 ?	—	35·7	28·9	70·9	71·9	72·5	98·5	97·8	−1·0	58·7	80·9	64·7	52·6	52·9	52·2	—	31·7	74·8	76
—	—	42·9	31·0	67·0	72·4	72·9	92·5	91·8	−5·4	62·8	72·3	66·4	57·9	57·1	56·2	—	30·2	75·2	77
41·0	10·5	37·5	29·5	68·5	68·8	70·9	99·6	96·6	−0·3	62·4	78·7	65·8	53·3	52·7	53·0	—	34·7	77·4	83
40·0	11·6	35·2	31·2	74·5	72·4	72·9	102·9	102·2	2·1	61·3	88·6	73·3	55·1	54·3	54·1	—	29·0	83·2	86
—	—	38·5	37·5	69·0	—	72·2	—	95·6	—	61·6	97·4	78·4	46·5	46·8	45·9	—	43·3 ?	80·5	76
39·6	14·1	39·5	28·7	72·0	72·3	73·9	99·6	97·5	−0·3	59·2	72·6	73·7	46·6	47·6	46·7	—	42·8	79·0	80
39·5	—	37·8	35·9	72·5	—	74·6	—	97·2	—	61·0	95·0	69·7	52·0	52·9	52·4	48·9	49·5	78·2	77
—	—	36·9 ?	—	—	—	—	94·2 ?	—	—	59·2	—	—	—	—	—	—	—	—	—
38·0	17·1	32·1	26·9	—	—	74·0	—	—	—	—	83·8	78·7	51·2	50·6	50·9	70·4	42·4	75·1	69
46·7	12·8	35·3	32·4	71·5	76·3	78·2	93·8	91·4	−4·7	60·2	91·8	63·6	57·3	57·1	56·9	—	30·5	72·5	73
—	—	36·3	29·2	74·6	75·9	78·1	98·2	95·4	−1·3	59·3	80·4	75·1	45·9	45·5	46·1	—	64·8	82·1	79
39·2	—	37·6	—	66·0	—	—	—	—	—	58·1	—	72·9	—	50·9	50·5	—	—	76·4	—
45·3	19·9 ?	36·2	31·5	69·4	68·9	70·4	100·7	98·5	0·5	56·7	87·0	73·5	50·2	52·0	50·0	50·2	31·8	73·9	78
46·5	13·8	40·0	34·0	74·3	70·5	71·8	105·5	103·6	3·9	59·2	85·0	70·2	56·7	56·5	56·9	47·6	36·1	77·5	74
—	—	—	—	69·1	—	—	—	—	—	59·2	—	71·0	—	53·2	52·2	—	—	74·7	—
34·0 ?	—	29·7	26·6	76·3	73·1	73·1	104·3	104·3	3·2	58·6	89·6	66·7	49·7	49·0	50·0	—	45·0	78·0	85
—	—	—	—	72·3	—	—	—	—	—	59·2	—	—	—	—	—	—	—	—	—
36·9	13·1	36·8	29·0	77·8	73·6 ?	74·7	105·7 ?	104·1	4·2 ?	62·4	78·8	72·9	49·5	49·7	49·6	—	34·8	82·7	84
35·0	—	34·3	29·5	77·5	70·5	71·4	109·9	108·7	7·0	57·5	86·0	—	54·5	54·0	54·7	—	—	83·9	—
38·9	11·1	37·8	29·8	72·5	74·2	75·3	97·8	96·3	−1·6	62·3	78·8	75·7	53·1	52·8	53·2	—	21·0	72·2	71
—	—	34·7	27·9	66·8	—	—	—	—	—	59·1	80·4	—	—	—	—	—	—	—	—
36·5	12·0	32·3	27·7	75·1	72·7	72·9	103·3	103·0	2·4	67·0 ?	85·8	75·5	54·2	54·1	54·1	56·5 ?	—	73·9	73
36·0	10·5	37·5	29·1	76·6	72·9	74·9	105·9	102·3	3·7	61·4	77·6	74·8	56·1	56·6	56·3	37·2	30·8	84·3	83
39·5	11·2	34·3	30·1	80·5	78·2	78·7	102·9	102·2	2·3	61·9	87·5	71·9	47·7	48·0	48·2	—	27·0	75·0	—
38·3	11·3	36·6	30·5	—	—	73·8	—	—	—	60·3	83·3	74·8	—	49·7	49·2	—	—	85·4	85
32·2 ?	—	38·4	29·1	—	74·2 ?	—	—	—	—	64·2	75·8	80·4	—	47·3	47·2	—	—	—	—
—	—	36·1	29·3	73·7	68·2 ?	—	108·2 ?	—	5·6 ?	59·4	81·2	—	—	—	—	—	—	—	—
40·0	14·2	34·0	27·0	73·9	70·1	71·7	105·3	103·0	3·8	59·0	79·4	66·7	64·9	62·9	64·3	—	38·6	67·9	68
40·0	11·3	—	—	73·3	—	—	—	—	—	60·8	—	69·8	51·1	50·2	51·7	—	—	80·7	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
36·0	—	—	—	68·2	74·4 ?	75·2	91·6 ?	90·7	−6·2 ?	62·4	—	78·0	—	47·5	47·4	—	—	—	85
33·6	13·6	—	—	78·0	—	—	—	—	—	—	—	68·2	51·4	51·2	50·8	32·9	31·1	82·3	78
—	—	38·4	29·2	68·4	—	69·2	—	98·9	—	57·5	76·0	68·7	47·3	47·2	46·6	—	33·0	82·2	81
38·1	14·8 ?	33·4	28·1	75·7	77·7	78·0	97·4	97·0	−2·0	58·1	84·1	65·4	61·9	60·0	61·7	41·3	43·3	82·3	82
39·4 ?	—	—	—	71·1	71·7	—	99·2	—	−0·6	—	—	75·1	51·4	51·5	51·2	—	—	91·1	90
35·7 ?	—	39·1	34·6	69·7	75·1	74·6	92·8	93·3	−5·4	62·8	88·5	72·4	49·6	47·8	49·5	—	43·1	82·5	81
40·9	—	—	—	71·0	—	—	—	—	—	61·1	—	72·9	52·5	52·4	53·1	—	36·6	68·0	76
—	—	34·2	28·0	67·5	—	69·5	—	97·7	—	58·1	81·9	69·7	52·7	53·3	52·7	37·9 ?	29·3	73·6	77
41·5	12·5	42·8	35·1	70·3	69·5	69·8	101·1	100·7	0·8	60·5	82·0	—	—	—	—	48·2	24·4	71·8	77
—	—	—	—	—	—	—	—	—	—	58·8	—	66·1	—	—	—	—	—	83·3	—
38·2	12·7	34·5	31·5	76·8	74·0	74·6	103·8	103·0	2·8	60·5	91·3	81·7	44·1	44·5	45·1	—	41·4	83·1	81
36·0	15·2 ?	35·1	—	70·0	66·8	67·4	104·8	103·9	3·2	58·2	—	85·1	49·5	50·8	50·1	—	21·9	87·3	90
39·1	15·2	33·4	26·6	71·3	71·3	74·2	100·0	96·2	0·0	58·4	79·6	78·5	53·2	54·4	53·8	—	49·3	79·5	78
—	—	—	—	76·5 ?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	34·9	27·5	73·7	73·4 ?	74·2	100·4 ?	99·3	0·3 ?	60·0	78·8	68·7	53·3	53·7	53·8	—	27·7	81·5	82
35·0	—	35·7	30·0	78·3	—	69·8	—	112·2	—	60·1	84·0	71·6	52·9	52·1	52·7	73·5	20·9	83·1	81
—	—	33·8	27·7	69·8	72·1	73·2	96·9	95·4	−2·2	58·4	81·9	67·7	58·5	59·0	58·6	—	35·7	81·4	82
40·5	—	35·1	29·5	75·4	73·2	74·3	103·0	101·5	2·2	57·5	84·0	—	—	—	—	—	—	78·5	—
—	—	35·5	29·1	68·5	66·6	67·7	102·9	101·2	1·9	59·4	81·9	—	—	—	—	—	—	—	76
41·5	12·1 ?	—	—	74·2	—	—	—	—	—	67·2	—	71·8	—	49·8	50·0	—	—	77·9	—
40·1	18·3	—	—	72·4	—	—	—	—	—	61·7	—	80·6	49·2	50·2	49·8	—	38·9	78·5	81
—	—	—	—	70·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	45·0	30·0	70·4	69·6	—	101·1	—	0·8	61·4	66·7	—	—	—	—	—	—	—	—
37·6	11·1	34·8	29·9	76·4	77·5	77·7	98·6	98·2	−1·1	65·3	85·9	64·3	50·5	51·3	51·1	—	15·0	76·7	75
—	—	—	29·0	76·9	75·3	—	102·2	—	1·6	70·4	—	—	—	—	—	—	—	—	—
41·5	13·9	31·5	25·1	77·1	73·5	75·7	104·1	101·8	3·6	59·5	79·7	73·2	51·3	51·7	52·1	—	34·5	82·8	81
—	—	—	—	72·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
36·0	13·0	36·9	33·2	75·6	73·1	73·9	103·4	102·2	2·5	62·8	89·0	72·2	51·0	52·3	52·5	61·5	45·6	86·1	81
42·4	—	—	—	74·6	—	—	—	—	—	—	—	72·0	55·5	55·5	55·6	52·3	27·0	—	72
37·8 ?	—	38·8	29·9	72·5	70·7	70·1	102·7	103·5	1·9	58·1	77·1	69·2	50·3	50·2	50·1	—	26·6	85·2	81
—	—	—	30·0	75·9	75·6	75·6	100·4	100·4	0·3	63·5	—	66·0	50·3	51·5	51·1	—	36·9	75·7	76
—	—	39·5	32·5	75·6	75·3	76·4	100·4	98·9	0·3	63·5	82·3	71·6	52·4	51·8	52·3	—	33·3	84·0	82
38·0 ?	—	—	30·8	73·0	75·6	77·8	96·7	93·9	−2·5	62·7	—	80·1	49·0	49·9	48·6	—	29·8	85·4	82

100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{O_1}, L$	100 $\frac{O_2}{O_1'}, R$	100 $\frac{O_2}{Lacr.\ O_1}, R$	100 $\frac{G_2}{G_1'}$	100 $\frac{G_2}{G_1}$	100 $\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alv.* $P\angle$	*Pros.* $P\angle$	$\theta_1\angle$	$\theta_2\angle$	
—	83·7	85·4	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture c
—	80·0	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture b
52·1	80·9	80·9	83·0	—	—	—	—	66°·7	74°·6	38°·7	—	83°	—	—	All sutures close
36·1	73·3	73·8	78·3	—	77·4 ?	—	26·3	—	—	—	84°	82°	—	—	Coronal and sagi
24·3	84·3	78·9	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagi
—	79·7	79·6	—	—	94·3	87·0	34·1	64°·1	72°·8	43°·1	—	81°	—	—	Third molars not
46·0	80·5	82·7	—	—	—	—	—	68°·0	67°·3	44°·7	75°	73°	37°·0	7°·7	Teeth worn and
31·7	74·8	76·5	—	—	95·7 ?	81·6 ?	—	64°·0	78°·0	38°·0	—	—	—	—	Coronal and sagi
30·2	75·2	77·2	—	—	—	—	—	67°·8	78°·9	33°·3	—	81°	—	—	Sagittal suture c
34·7	77·4	83·2	—	—	94·7	82·8	25·6	64°·5	78°·2	37°·3	89°	86°	26°·5	10°·8	Coronal and sagi
29·0	83·2	86·5	—	—	80·0	72·3	29·0	67°·0	69°·4	43°·6	79°·5	77°	33°·5	10°·1	Coronal and sagi
43·3 ?	80·5	76·2	—	—	—	—	—	65°·6	73°·5	41°·0	—	—	—	—	Coronal and sagi
42·8	79·0	80·2	—	—	82·7	76·4	35·6	59°·0	78°·2	42°·8	—	85°	—	—	All sutures partl
49·5	78·2	77·3	81·7	86·7	78·2	70·7	—	64°·3	74°·9	40°·9	82°	80°	33°·7	7°·1	Coronal and sagi
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture b
42·4	75·1	69·0	76·7	82·5	83·1	74·8	45·0	62°·3	75°·0	42°·6	85°	83°	32°·7	10°·0	All sutures oblit
30·5	72·5	73·4	75·7	79·7	110·0	98·7	27·4	60°·9	80°·1	39°·0	84°	82°	35°·1	3°·9	Coronal and sagi
64·8	82·1	79·6	86·9	—	—	—	—	56°·0	84°·0	40°·0	—	90°·5 ?	—	—	Coronal and sagi
—	76·4	—	—	—	—	—	—	—	—	—	—	—	—	—	
31·8	73·9	78·1	78·1	79·8	89·0	79·5	43·9 ?	65°·0	75°·0	40°·0	84°·5	83°	30°·5	9°·5	All sutures close
36·1	77·5	74·7	77·7	—	101·3	93·0	29·7	64°·2	74°·8	40°·9	87°	87°	28°·8	12°·2	Sagittal and lam
—	74·7	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures close
45·0	78·0	85·0	—	—	77·3 ?	67·2 ?	—	65°·5	75°·0	39°·5	83°	81°	31°·5	8°·0	Sagittal and lam
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sutures closed, c
34·8	82·7	84·8	87·0	88·6 ?	76·9	71·9	35·5	66°·0	69°·5	44°·5	82°	80°	32°·0	12°·5	Coronal and sagi
—	83·9	—	—	—	—	—	—	65°·1	66°·3	48°·6	—	—	—	—	Sagittal suture c
21·0	72·2	77·9	—	—	73·1	66·5	28·5	70°·8	72°·0	37°·2	—	77°	—	—	Coronal suture b
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures oblit
—	73·9	77·7	80·8 ?	—	84·9	75·9	32·9	63°·0	75°·9	41°·1	89°	87°	28°·0	13°·1	Sagittal and lam
30·8	84·3	82·1	88·3	—	80·7	76·3	29·2	—	—	—	—	—	—	—	Sagittal suture c
27·0	75·0	—	—	—	89·8	80·1	28·3	62°·3	77°·8	39°·9	83°·5	79°	34°·2	5°·7	Teeth worn. JR
—	85·4	83·5	88·3	—	—	—	29·5	60°·8	77°·0	42°·0	—	88°	—	—	Coronal suture p
—	—	—	—	—	—	—	—	63°·8	74°·6	41°·8	—	—	—	—	Coronal and sagi
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture c
38·6	67·9	68·4	75·0	—	86·2	76·8	35·5	69°·1	71°·3	39°·6	80°	78°	30°·9	8°·7	Sagittal suture b
—	80·7	—	—	—	—	—	28·2	—	—	—	—	—	—	—	Coronal suture b
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	
—	—	84·1 ?	—	—	—	—	—	62°·0	76°·5	41°·5	—	—	—	—	Teeth worn. JL
31·1	82·3	79·4	84·5	86·5	76·9	71·5	40·5	—	—	—	85°·5	84°	—	—	
33·0	82·2	83·9	—	—	—	—	—	63°·0	78°·0	39°·0	85°	83°·5	32°·0	7°·0	Coronal suture o
43·3	82·3	80·6	86·4	86·7	86·2	78·9	38·8 ?	65°·8	75°·9	38°·3	89°	86°·5	25°·2	13°·1	Teeth worn. JL
—	91·1	90·0	—	—	85·5 ?	79·3 ?	—	63°·0	68°·0	49°·0	83°	81°·5	34°·0	15°·0	Coronal suture p
43·1	82·5	81·9	—	—	73·3 ?	63·4 ?	—	64°·0	75°·9	39°·9	81°	80°·5	35°·0	5°·1	Sagittal suture o
36·6	68·0	72·2	—	—	—	—	—	—	—	—	82°	80°	—	—	Coronal and sagi
29·3	73·6	75·2	76·6	—	—	—	—	65°·2	76°·9	37°·9	—	82°	—	—	Sagittal suture c
24·4	71·8	70·4	73·8	77·5	—	—	30·1	—	—	—	—	—	—	—	Teeth worn. JR
—	83·3	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture b
41·4	83·1	84·9	—	—	74·7	69·4	33·2	62°·0	73°·2	44°·7	—	73°	—	—	Sagittal suture c
21·9	87·3	90·9	—	—	72·7	69·2 ?	42·2	64°·7	66°·9	48°·4	85°·5	83°·5	29°·8	18°·6	Coronal suture c
49·3	79·5	79·0	—	—	76·8	71·1	38·9	65°·0	73°·9	41°·1	86°	82°·5	29°·0	12°·1	Coronal and sagi
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	
27·7	81·5	83·1	85·1	87·5	—	—	—	70°·7	70°·6	38°·7	—	—	—	—	Coronal suture o
20·9	83·1	83·3	85·6	—	79·5	74·1	—	60°·2	77°·8	41°·9	—	87°	—	—	All sutures oblit
35·7	81·4	83·6	—	—	—	—	—	66°·2	74°·0	39°·8	—	79°?	—	—	All sutures close
—	78·5	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal and cor
—	—	79·1	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture c
—	77·9	—	—	—	—	—	29·2 ?	66°·0	73°·0	41°·0	—	—	—	—	Coronal and sagi
38·9	78·5	80·2	—	—	80·8	71·0	45·6	—	—	—	81°	78°·5	—	—	
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures close
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	JR.
15·0	76·7	74·0	83·4	85·0	82·8	80·0	29·5	67°·4	75°·2	37°·4	83°·5	80°	29°·1	8°·3	J=.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagi
34·5	82·8	80·8	—	—	86·1	74·4	33·5	67°·2	72°·7	40°·1	81°	79°·5	31°·8	8°·3	J=.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures oblit
45·6	86·1	88·5	92·2	92·2	75·5	70·3	36·1	64°·1	73°·9	42°·0	81°	79°	34°·9	7°·1	JL.
27·0	—	75·9	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture o
26·6	85·2	82·1	87·5	—	85·1 ?	75·0?	—	65°·0	74°·2	40°·8	83°	80°	32°·0	8°·8	JR.
36·9	75·7	76·1	—	—	—	—	—	66°·6	75°·9	37°·5	—	79°	—	—	Sagittal suture b
33·3	84·0	84·9	—	—	—	—	—	63°·0	74°·9	42°·1	84°	81°	33°·0	9°·1	Coronal suture c
29·8	85·4	81·7	—	—	71·4 ?	64·9 ?	—	71°·0	68°·7	40°·3	77°·5	74°	31°·5	8°·8	Coronal suture c

Remarks
Coronal suture closed and partly obliterated, sagittal partly closed. Teeth worn. JR.
Coronal suture beginning to close. JR.
All sutures closed, coronal partly obliterated. 3 teeth lost before death. JL.
Coronal and sagittal sutures closed. Teeth markedly worn.
Coronal and sagittal sutures beginning to close. All, or nearly all, teeth lost before death. JR.
Third molars not erupted. JR.
Teeth worn and 3 lost before death. JR.
Coronal and sagittal sutures beginning to close. Teeth markedly worn. J=.
Sagittal suture closed, lambdoid partly closed. Teeth worn and 10 or more lost before death. J=.
Coronal and sagittal sutures beginning to close. Teeth worn. J=.
Coronal and sagittal sutures partly obliterated. JR.
Coronal and sagittal sutures obliterated, lambdoid closed. Teeth worn and 1 lost before death. J=.
All sutures partly closed and obliterated. Teeth worn. JR.
Coronal and sagittal sutures obliterated, lambdoid closed. Teeth worn. JL.
Coronal suture beginning to close.
All sutures obliterated. Teeth worn. JR.
Coronal and sagittal sutures beginning to close. JR.
Coronal and sagittal sutures beginning to close. Teeth worn and 3 lost before death. JR.
—
All sutures closed. Teeth worn. JR.
Sagittal and lambdoid sutures obliterated, coronal closed. Teeth worn. JR.
All sutures closed and nearly obliterated. 1 tooth lost before death.
Sagittal and lambdoid sutures closed, coronal closing. Teeth worn. JR.
Sutures closed, coronal partly obliterated.
Coronal and sagittal sutures beginning to close. Teeth markedly worn. JR.
Sagittal suture closed, coronal closing. Teeth worn and 3 lost before death. JR.
Coronal suture beginning to close. JR.
All sutures obliterated. JR.
Sagittal and lambdoid sutures obliterated, coronal closed. Teeth worn. JR.
Sagittal suture closed.
Teeth worn. JR.
Coronal suture partly obliterated, sagittal and lambdoid closed. JL.
Coronal and sagittal sutures closed. Teeth worn and 1 molar lost before death. JR.
Sagittal suture closed, coronal and lambdoid closing.
Sagittal suture beginning to close. Teeth worn. JR.
Coronal suture beginning to close.
—
Teeth worn. JL.
—
Coronal suture obliterated, sagittal partly obliterated, lambdoid closed. 9 or more teeth lost before death. JR.
Teeth worn. JL.
Coronal suture partly obliterated, sagittal closed.
Sagittal suture obliterated, coronal and lambdoid closed. 2 teeth lost before death. JR.
Coronal and sagittal sutures beginning to close.
Sagittal suture obliterated, coronal partly obliterated, lambdoid partly closed. Several teeth lost before death. JR.
Teeth worn. JR.
Coronal suture beginning to close.
Sagittal suture closed, coronal beginning to close. JR.
Coronal suture closed, sagittal beginning to close. Teeth worn, 1 molar lost before death. JR.
Coronal and sagittal sutures closed, lambdoid beginning to close. Teeth worn. JR.
—
Coronal suture obliterated, sagittal closed. Teeth worn and 4 molars lost before death. JR.
All sutures obliterated. Teeth worn and 1 lost before death. JL.
All sutures closed, coronal and sagittal partly obliterated. 5 or more teeth lost before death. JR.
Sagittal and coronal sutures beginning to close. JR.
Sagittal suture closed and partly obliterated, coronal closing. JR.
Coronal and sagittal sutures partly closed, lambdoid closing.
—
All sutures closed, coronal obliterated.
JR.
J=.
Coronal and sagittal sutures beginning to close. J=.
J=.
All sutures obliterated.
JL.
Sagittal suture obliterated, coronal partly obliterated, lambdoid closed.
JR.
Sagittal suture beginning to close. 2 molars lost before death. JR.
Coronal suture closed, sagittal and lambdoid partly closed. 3 or more teeth lost before death. JR.
Coronal suture closed, sagittal closing. 1 molar lost before death. JL.

To face p. 284.

Serial No.	Grave Mark	Probable Dynasty	C	B′	B″	B	Bias-terionic B	F	L	H′	H	OH	LH
148	XB/LJ	XIII	—	96·7	—	128·5	101·0	178	179	124	124	108	92
149	XB/CB	„	1263	96·0	108	131·5	100·5	181	180·5	128	127·5	105	93
150	XB/GC	„	—	89·7	—	128·5	100·3	179	178·5	128·5	130·5	111	99
151	XB/UL	„	—	91·5	—	—	—	180	178	—	—	—	—
152	XB/AC	„	—	97·4	107	123	106·6	180	178·5	131	130·5	115	99
153	XB/RL	„	1185	91·5	107	134·5	100·5	171	171·5	119·5	121·5	104	91
154	XB/R	„	—	91·8	—	126	95·2 ?	184	184	—	—	111	—
155	XB/PB	„	1372	92·0	110	137	104·2	179	177·5	—	—	106	—
156	XB/XA	„	—	95·5	—	130·5	106·0	184	183·5	131	131	109	102
157	XB/XA	„	—	87·3	—	134·5	102·1	192	191·5	—	—	—	98
158	XB/UB	„	1083	83·0	103·5	126·5	94·0	168·5	169	125·5	126	106	96
159	XB/X	„	—	95·0	—	130·5	100·0	182	182	—	—	—	—
160	1084 B	„	—	87·4	111	130	95·0	163	165	124·5	124	106	93
161	15 B/B	XII	1094	84·4	105·5	129	96·1	167·5	166·5	125	124·5	101	94
162	8 A/B	„	—	87·0	—	126·5	—	178	178	124	—	105·5	98
163	2321 B	XIII	—	97·5	115	132	98·0	176·5	176	125	127	106·5	96
164	1079 B	„	1285	86·0	108·5	134	101·0	175	176	129	132	110	94
165	2354 B	„	1270	92·2	112	139	98·0	176	175·5	127	125·5	105	—
166	1096 B	„	1167	89·1	111	128	94·0	170	170	124	125	107	95
167	41 B	XII	—	82·8	—	124	96·0	170	172	—	—	102	—
168	1085 B	XIII	1431	92·3	117	133	100·0	184	185	139	137·5	113	103
169	1067 B	„	—	95·0	—	128	100·2	179·5	180	126·5	—	—	102
170	43 B	XII	—	87·2	—	125·5	105·2	172	172	—	—	105	—
171	47 B	„	—	92·7	111	136	99·0	168	169	120	121·5	108	90
172	339 B	„	1280	84·0	106	129·5	98·0	181	179·5	124	126·5	107·5	91
173	1090 B	XIII	1231	84·8	111	135	102·0	170	171·5	131	132·5	108	95
174	1604	„	1310	93·0	—	136	100·4	180	180	124	124	107	95
175	332 B	XII	—	86·5	—	125	—	180	179	128	—	—	—
176	1058 B	XIII	—	—	—	127	—	178	180	—	—	—	—
177	334 B	XII	1154	87·0	103·5	121·5	95·7	175	173·5	123	123	103	89
178	1045 B	XIII	—	85·0	100	125	93·3	173	173	128	129	106·5	95
179	1805 B	„	1309	90·3	107	136	102·0	174·5	175	131	131·5	107	97
180	14 B	XII	—	98·8	—	131	99·3	179	177·5	140	140	119	100
181	1026 (a) B	XIII	—	90·9	—	130·5	101·5	183	182·5	128	—	—	100
182	III 308	XII	—	86·7	—	128	95·0	176	176	—	—	—	—
183	III 301 (D)	„	—	88·6	105	120	101·0	177	175·5	128	—	—	92
185	IV 415 (A)	„	—	79·0	103·5	127	95·3	169·5	168·5	124·5	123	104	95
186	415 (B)	„	—	93·4	—	133	107·0	170·5	170	122	121	106	94
187	434 (B)	„	1136	89·0	107	127·5	95·0	166	166·5	124	122	103	96
188	420	„	—	83·1	—	127·5	99·2	173	174	127	—	107	99
189	1067 (i)	XIII	—	94·3	—	—	—	187	187·5	—	—	—	—
190	1067 (ii)	„	—	90·4	—	131	—	180	180·5	129	128	105	99
191	1068	„	—	83·2	—	132	98·0 ?	184	185·5	—	—	102	—
192	1805 B	„	—	82·2	—	128 ?	96·0	167·5	168	121·5	122	103·5	95
193	1049 B	„	—	89·0	109	133·5	97·0	172	172	122·5	123	105	97
194	1025	„	—	93·2	107	125	100·0	174·5	175·5	129	131	107	96
195	1024 *b*	„	—	90·6	—	126	97·3	185	185	—	—	109	—
196	450 A	XII	—	98·0	117	126	92·7	181	179	124	—	—	99
197	436	„	—	87·2	105	130	99·1	168	169·5	126	127	105	91
198	1044	XIII	—	87·0	—	131·5	—	181	179·5	—	—	—	—
199	435	XII	1256	93·1	110	124·5	98·1	177	177	133·5	132·5	109	97
200	1026	XIII	1280	87·8	—	127·5	100·0	179	178	129	129	102·5	97
201	1603	„	1128	80·5	105	121·5	100·2	170	169·5	128	129	106	95
202	1044	„	—	93·2	113	133	97·1	174	173	129	128	112	92
203	3501 C	„	1182	82·8	107	136·5	97·0	172	171·5	125	125	107	95
204	425 (i)	XII	1167	79·3	101	127·5	105·4	165·5	166	124·5	124·5	102	92
206	452 A	„	1238	84·5	100	129·5	104·1	178	177	120	121	102	93

OH	*LB*	S_1'	S_2'	S_3'	S_1	S_2	S_3	*S*	*U*	*Q'*	*G'H*	*GL*	*GB*	*J*	*NH'*	*NH, R*	*NH, L*	*NB*	*DS*
108	92·0	114·2	109·5	93·9	133	119	110	362	500	288	70·0	90·8	95·1	127	49·0	49·0	49·3	24·0	—
105	93·0	111·3	117·4	96·9	129	129	118	376	505	285	66·5	86·1	90·0	121·5 ?	46·7	46·9	46·8	25·2	—
111	99·8	108·3	113·0	94·0	123	127	114	364	497	294	62·3	98·7	91·4	115	44·9	44·6	45·0	24·3	—
—	—	112·2	113·5	—	131	130	—	—	—	—	69·3	94·0	93·5	—	50·2	51·0	51·1	—	—
115	99·0	112·1	111·4	90·5	131	122	108	361	492	295	66·4	102·4	93·2	113·5	42·4	41·9	42·5	22·3	13·0
104	91·2	97·7	110·9	94·8	111	126	114	351	489	288·5	63·4	88·0	94·8	121·5	46·2	46·8	47·2	27·3	—
111	—	114·3	110·0	91·1	129	120	109	358	495	293	70·5	—	—	—	—	—	—	—	—
106	—	107·3	109·0	96·3	123	118	121	362	505	295·5	69·5	—	88·6	—	49·2	—	—	22·1	—
109	102·0	102·6	115·3	92·1	127	130	113	370	506	296	71·3	93·9	91·7	123	52·1	52·3	52·3	24·9	—
—	98·5	—	—	—	—	—	—	389	518	—	—	—	—	—	—	—	—	—	—
106	96·1	106·4	100·4	89·3	121	109	106	336	468	293	61·5	93·5	91·4	—	42·6	42·6	42·6	22·6	—
—	—	109·3	116·0	97·7	126	130	116	372	506	—	—	—	—	—	—	—	—	—	—
106	93·1	100·6	104·0	93·3	112	113	109	334	461	292	62·0	87·4	92·0	—	43·7	45·0	44·0	23·0	—
101	94·2	102·5	101·1	90·7	118	112	109	339	471	282	65·0	93·0	87·0	115	46·0	46·0	46·0	25·0	—
105·5	98·0	108·2	110·1	91·5	125	126	110	361	487	282	64·5	99·0	98·0	121	45·8	46·5	46·3	26·2	—
106·5	96·1	101·4	115·7	—	117	131	—	—	495	293	67·4	91·2	85·5	123	48·6	49·3	48·9	25·2	—
110	94·0	105·5	104·8	107·0	119	117	126	362	494	293	65·0	94·5	91·5	120	43·2	44·0	43·5	22·3	—
105	—	100·9	113·0	100·3	117	128	120	365	487	292	63·3	89·1	92·0	117	45·7	45·6	45·7	23·2	—
107	95·1	98·5	105·3	91·7	118	119	107	344	479	292	64·0	94·2	84·5	120·5	50·0	49·8	50·2	22·3	—
102	—	106·7	102·6	—	121	112	—	—	470	271·5	66·3	—	87·6	118	46·1	46·4	46·2	26·0	—
113	103·0	115·6	112·8	98·6	132	123	121	376	498	313	63·0	90·2	96·0	—	46·0	46·2	46·3	22·8	17·8
—	102·6	108·8	111·0	91·7	123	124	111	358	495	—	70·6	—	89·0	—	45·4	46·2	46·0	27·0	—
105	—	99·5	111·1	—	113	127	—	—	482	285·5	70·1	—	91·8	—	50·7	49·4	50·9	27·5	—
108	90·3	103·0	105·1	93·5	118	118	110	346	487	296	60·5	92·7	90·4	115	45·0	45·3	45·9	22·6	—
107·5	91·1	109·8	118·0	88·7	130	138	106	374	489	288	60·4	89·8	84·4	109	42·0	42·7	42·5	24·2	—
108	95·4	103·7	108·9	95·9	119	122	111	352	485	298	58·6	95·8	81·7	115	42·0	42·0	42·0	24·3	—
107	95·2	110·8	111·9	94·8	129	123	111	363	501	297	71·0	94·0	93·0	118·5	51·2	52·0	51·7	25·7	17·4
—	—	111·5	114·5	90·0	127	129	111	367	—	—	—	—	—	—	—	—	—	24·7	—
—	—	106·8	111·3	—	121	124	—	—	—	—	—	—	—	—	—	—	—	—	—
103	89·7	106·5	112·7	93·2	124	125	112	361	477	281	62·5	92·0	83·6	114	44·0	44·0	44·3	23·2	—
106·5	95·6	106·3	107·0	92·2	121	120	111	352	475	284	67·9	90·5	87·0	—	48·0	48·9	49·7	20·8	—
107	97·9	108·8	109·4	93·5	124	124	112	360	478	292·5	63·2	93·6	99·7	120	45·1	45·7	45·6	28·2	—
119	100·0	119·1	108·4	103·9	137	119	123	379	493 ?	313	68·0	100·4	91·6	—	46·1	48·0	46·3	25·5	—
—	100·2	109·6	109·8	98·5	127	122	117	366	502	—	—	—	—	—	—	—	—	—	—
—	—	102·0	113·2	—	115	130	—	—	483	—	—	—	—	—	—	—	—	—	—
—	92·1	102·5	116·3	93·1	122	128	112	362	483	—	—	—	—	—	—	—	—	—	—
104	95·1	99·3	110·2	89·0	115	122	106	343	469	285	60·6	93·3	86·1	117	44·0	44·0	44·1	22·3	—
106	94·4	102·3	106·9	96·1	116	117	113	346	487	303	—	—	85·7	113	48·9	50·3	50·0	28·7	—
103	96·9	101·5	102·2	87·7	117	112	104	333	466	283	62·8	92·5	93·2	120	47·3	47·1	48·0	24·1	—
107	99·7	108·9	106·6	91·7	122	119	110	351	477	294	72·8	99·3	90·0	—	55·3 ?	54·8 ?	55·4 ?	19·7 ?	—
—	—	107·8	116·3	90·2	124	128	106	358	—	—	68·0	105·4	89·3	—	46·1	46·5	46·5	27·8	11·1
105	99·1	108·1	100·3	101·9	122	107	127	356	499	287	69·0	97·2	93·8	122	46·5	46·1	46·7	28·1	—
102	—	115·7	110·6	101·7	134	123	123	380	498	297	64·8	—	88·6	—	46·1	46·3	46·3	22·8	11·3
103·5	95·2	100·2	109·0	89·1	113	124	103	340	465	279	69·0	96·7	94·0	117	47·0	47·5	47·6	20·1	—
105	97·3	107·0	100·6	92·1	124	112	110	346	485	297	67·2	100·9	90·2	119	46·0	47·2	46·1	27·2	9·1
107	96·1	103·7	108·6	99·0	117	122	117	356	480	284	—	—	—	—	—	—	—	—	—
109	—	114·8	119·1	90·7	131	134	105	370	501	287	70·0	—	90·2	—	46·1	46·9	46·6	26·7	—
—	99·4	107·4	110·3	91·1	124	123	108	355	497	—	—	—	—	—	—	—	—	—	—
105	91·5	101·0	112·6	89·0	114	127	105	346	471	286·5	63·1	86·4	86·8	—	46·3	46·0	46·4	22·1	—
—	—	107·5	103·9	102·1	124	114	132	370	—	—	57·0	—	91·6	—	42·0	43·2	42·0	24·8	—
109	97·3	108·2	105·7	100·0	124	114	123	361	485	295	62·1	94·3	92·3	121	45·0	46·2	45·3	24·9	9·9
102·5	97·0	106·1	116·5	93·2	123	132	109	364	486	285	63·8	87·8	91·7	—	47·0	47·5	47·0	25·2	—
106	95·0	112·5	—	—	125	—	—	344	466	280	68·6	97·2	91·2	112·5	46·8	46·9	46·7	25·4	10·1
112	92·1	103·4	118·1	91·6	121	133	111	365	490	286	61·3	88·5	89·9	—	—	43·7	43·6	24·4	—
107	95·1	104·5	99·1	100·3	119	108	121	348	477	283	—	—	94·7	—	45·0	46·0	45·2	28·2	—
102	92·2	102·5	106·6	89·3	117	121	105	343	467	283	59·2	86·0	83·0	117	45·2	44·6	45·3	21·2	—
102	93·0	102·4	106·8	94·3	121	120	118	359	482	—	63·5	90·9	85·8	112	43·7	43·9	44·0	24·3	—

TABLE IX. *Individual Measurements o*

L	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	O_1R	O_1L	$O_1'R$	*Lacr.* $O_1'R$	O_2R	O_2L	G_1'	G_1	G_2	*EH*	*fml*
3	24·0	—	—	—	2·3	11·3	41·2	43·0	—	—	36·7	36·0	—	—	—	—	43·0
3	25·2	—	—	—	—	9·9	41·4	41·9	39·1	38·0	34·0	35·0	—	—	—	—	32·6
0	24·3	—	—	—	4·4	12·7	40·3	40·3	38·6	37·9	33·2	33·2	45·2	47·9	32·2	—	34·5
1	—	—	—	—	—	—	—	—	—	—	—	—	43·0	46·2	39·9	10·3	—
5	22·3	13·0	23·8	38·0	4·7	15·2	41·6	42·4	37·8	36·2	30·8	30·9	51·9	56·6	40·2	15·2	38·0
2	27·3	—	—	—	4·1	9·4	40·7	41·6	—	—	35·5	35·8	40·2	44·1	—	—	30·4
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	22·1	—	—	—	—	—	—	—	—	—	—	—	42·0	44·8	31·0	13·0	36·3
3	24·9	—	—	—	1·8	8·4	44·0	43·1	—	—	34·0	34·0	44·1	—	39·0	12·8	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·8
5	22·6	—	—	—	—	—	37·3	37·9	—	—	32·0	31·0	—	—	—	—	36·2
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0	23·0	—	—	—	3·7	13·9	37·6	37·9	35·6	35·6	32·0	31·6	43·3	47·9	37·6	10·6	33·5
0	25·0	—	—	—	3·9	10·8	37·6	37·8	36·3	35·2	31·1	31·0	44·0	48·2	38·3	14·4	34·0
3	26·2	—	—	—	1·7	11·2	40·0	40·0	—	—	31·2	31·2	—	—	—	—	34·4
9	25·2	—	—	—	3·3	10·4	42·7	41·9	—	—	34·4	33·0	—	—	37·2	—	—
5	22·3	—	—	—	—	—	40·0	39·7	—	—	31·2	31·0	47·2	50·1	—	—	35·6
7	23·2	—	—	—	2·0	7·1	39·0	40·3	—	—	33·0	32·5	—	—	37·3	—	32·3
2	22·3	—	—	—	6·1	9·6	37·4	39·3	36·0	35·7	30·0	28·9	44·2	49·7	37·8	11·5	33·3
2	26·0	—	—	—	2·5	10·1	40·3	39·2	38·0	—	31·2	32·5	—	—	—	—	—
3	22·8	17·8	22·8	38·5	4·9	10·9	41·0	41·7	39·0	39·0	33·2	32·0	44·0	48·9	38·8	12·1	35·4
0	27·0	—	—	—	—	—	40·7 ?	39·8 ?	—	—	—	32·9 ?	52·1	56·6	—	—	34·1
9	27·5	—	—	—	3·8	12·5	40·5	39·0	36·6	36·4	32·0	32·5	41·9	—	38·2	13·0	—
9	22·6	—	—	—	2·6	9·2	40·0	41·0	37·6	—	—	33·0	43·7	46·6	36·3	12·3	31·2
5	24·2	—	—	—	2·8	8·8	38·4	39·1	—	—	30·0	29·2	42·4	49·1	39·2	13·4	34·9
0	24·3	—	—	—	2·7	7·9	38·4	39·2	—	—	30·2	30·0	45·9	50·2	—	—	35·8
7	25·7	17·4	21·1	28·5	2·1	9·2	41·4	41·7	40·9	39·1	34·0	33·0	45·4	49·7	40·1	—	37·7
	24·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	33·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
3	23·2	—	—	—	2·0	7·8	37·5	37·0	—	—	32·0	31·8	45·9	48·8	37·0	11·9	34·0
7	20·8	—	—	—	—	—	40·9	—	—	—	—	—	—	—	36·4	11·0	32·3
6	28·2	—	—	—	2·3	9·1	40·0	40·2	—	—	32·0	33·0	45·7	49·2	41·6	11·2	33·4
3	25·5	—	—	—	1·1	8·2	41·7	39·6	—	—	33·7	35·0	49·0	—	—	—	33·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	33·0
1	22·3	—	—	—	2·6	8·8	37·6	38·1	36·0	35·0	29·7	29·3	44·8	47·7	—	—	32·0
0	28·7	—	—	—	3·0	9·9	40·0	41·1	—	—	37·2	37·2	—	—	35·8	—	35·3
0	24·1	—	—	—	3·9	9·9	40·3	39·7	35·7	35·7	30·1	30·4	46·3	51·0	35·6	—	36·1
4 ?	19·7 ?	—	—	—	—	—	40·2	—	—	—	34·7	—	—	—	41·7	—	33·2
5	27·8	11·1	24·0	37·5	3·7	13·7	41·9	40·1	40·0	38·2	32·6	33·9	—	—	33·2	15·2	—
7	28·1	—	—	—	3·0	9·3	41·2	41·0	—	—	34·7	36·0	—	—	—	—	36·7
3	22·8	11·3	19·9	32·0	—	—	36·7	37·1	34·9	—	32·9	33·0	46·0	49·9	—	—	—
6	20·1	—	—	—	1·4	6·1	40·4	39·9	38·8	37·0	34·0	33·9	48·1	56·5	—	—	34·0
1	27·2	9·1	22·4	32·0	3·8	12·0	39·7	40·0	37·1	36·0	33·0	33·6	48·8	52·2	37·0	12·7	32·1
	—	—	—	—	—	—	41·8	—	—	—	34·7	—	—	—	—	—	33·7
5	26·7	—	—	—	3·1	11·3	41·0	41·3	—	—	34·2	34·6	46·3	52·8	37·4	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·1
4	22·1	—	—	—	4·1	8·6	37·3	37·8	—	—	31·9	32·0	41·6	46·1	—	—	33·5
0	24·8	—	—	—	2·3	8·8	38·9	36·7	37·0	36·0	30·9	29·9	44·7	47·7	38·2	—	—
3	24·9	9·9	21·8	31·0	2·5	10·3	40·3	40·0	39·4	37·6	33·9	33·0	45·1	50·0	—	—	33·9
0	25·2	—	—	—	3·7	10·0	41·1	41·0	—	—	34·2	34·8	42·4	46·6	—	—	36·0
7	25·4	10·1	23·6	35·5	0·5	6·9	39·1	38·9	36·0	34·0	31·0	31·4	45·5	48·8	35·4	13·7	34·8
6	24·4	—	—	—	4·3	13·9	39·6	38·9	—	—	31·8	30·1	40·0	44·4	34·2	10·9	32·5
2	28·2	—	—	—	—	—	36·4	37·0	—	—	31·9	31·4	—	—	—	—	33·4
3	21·2	—	—	—	4·4	9·2	37·0	37·0	36·2	—	29·8	30·3	43·1	45·1	35·8	12·0	35·0
0	24·3	—	—	—	3·1	9·9	38·5	39·0	—	—	32·3	31·9	46·1	50·0	35·7	12·9	33·0

...urements of Female Kerma Skulls.

EH	*fml*	*fmb*	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{H}{L}$	100 $\frac{B}{H'}$	100 $\frac{B}{H}$	100 $\frac{(B-H')}{L}$	*Oc. I.*	100 $\frac{fml}{fmb}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH}$, *R*	100 $\frac{NB}{NH}$, *L*	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1}$, *R*	100 $\frac{O_2}{O_1}$, *L*	100 $\frac{O_2}{O_1'}$, *R*
—	43·0	31·1	71·8	69·3	69·3	103·6	103·6	2·5	62·5	72·3	73·6	49·0	49·0	48·8	—	20·4	89·1	83·7	—
—	32·6	—	72·9	70·9	70·6	102·7	103·1	1·9	58·8	—	73·9	53·9	53·7	53·8	—	—	82·1	83·5	86·9
—	34·5	25·3	72·0	72·0	73·1	100·0	98·5	0·0	59·1	73·3	68·2	54·1	54·5	54·0	—	34·6	82·4	82·4	86·0
10·3	—	—	—	—	—	—	—	—	—	—	74·1	—	—	—	—	—	—	—	—
15·2	38·0	27·0	68·9	73·4	73·1	93·9	94·3	−4·5	58·7	71·0	71·2	52·6	53·2	52·5	54·6	30·9	74·0	72·8	81·5
—	30·4	28·0	78·4	69·7	71·3	112·5	110·7	8·7	59·8	92·1	66·8	59·1	58·3	57·8	—	43·6	87·2	86·0	—
—	—	—	68·1	—	—	—	—	—	60·3	—	—	—	—	—	—	—	—	—	—
13·0	36·3	—	77·2	—	—	—	—	—	56·8	—	78·4	44·9	—	—	—	—	—	—	—
12·8	—	30·0	71·1	71·4	71·4	99·6	99·6	−0·3	58·2	—	77·7	47·8	47·6	47·6	—	21·4	77·3	78·9	—
—	36·8	29·7	70·2	—	—	—	—	—	—	80·7	—	—	—	—	—	—	—	—	—
—	36·2	28·3	74·9	74·3	74·6	100·8	100·4	0·6	61·0	78·2	67·3	53·0	53·0	53·0	—	—	85·8	81·8	—
—	—	—	71·7	—	—	—	—	—	56·4	—	—	—	—	—	—	—	—	—	—
10·6	33·5	25·9	78·8	75·4	75·1	104·4	104·8	3·3	67·8	77·3	67·4	52·6	51·1	52·3	—	26·6	85·1	83·4	89·9
14·4	34·0	27·1	77·5	75·1	74·8	103·2	103·6	2·4	59·9	79·7	74·7	54·3	54·3	54·3	—	36·1	82·7	82·0	85·7
—	34·4	28·8	70·8	69·3	—	102·0	—	1·4	59·9	83·7	65·8	57·2	56·3	56·6	—	15·2	78·0	78·0	—
—	—	—	75·0	71·0	72·2	105·6	103·9	4·0	—	—	78·8	51·9	51·1	51·5	—	31·7	80·6	78·8	—
—	35·6	29·9	76·1	73·3	75·0	103·9	101·5	2·8	61·9	84·0	71·0	51·6	50·7	51·3	—	—	78·0	78·1	—
—	32·3	29·0	79·2	72·4	71·5	109·4	110·8	6·8	60·3	89·8	68·8	50·8	50·9	50·8	—	28·2	84·6	80·6	—
11·5	33·3	30·9	75·3	72·9	73·5	103·2	102·4	2·3	63·0	92·8	75·7	44·6	44·8	44·4	—	63·5	80·2	73·5	83·3
—	—	—	72·1	—	—	—	—	—	—	—	75·7	56·4	56·0	56·3	—	24·7	77·4	82·9	82·1
12·1	35·4	30·0	71·9	75·1	74·3	95·7	96·7	−3·2	58·2	84·7	65·6	49·6	49·4	49·2	78·1	44·9	81·0	76·7	85·1
—	34·1	30·0	71·1	70·3?	—	101·2?	—	0·8?	59·4	88·0	79·3	59·5	58·4	58·7	—	—	—	82·7?	—
13·0	—	—	73·0	—	—	—	—	—	—	—	76·4	54·2	55·7	54·0	—	30·4	79·0	83·3	87·4
12·3	31·2	23·3	80·5	71·0	71·9	113·3	111·9	9·5	62·1	74·7	66·9	50·2	49·9	49·2	—	28·3	—	80·5	—
13·4	34·9	26·1	72·1	69·1	70·5	104·4	102·4	3·1	60·4	74·8	71·6	57·6	56·7	56·9	—	31·8	78·1	74·7	—
—	35·8	27·8	78·7	76·4	77·3	103·1	101·9	2·3	64·1	77·7	71·7	57·9	57·9	57·9	—	34·2	78·6	76·5	—
—	37·7	—	75·6	68·9	68·9	109·7	109·7	6·7	62·6	—	76·3	50·2	49·4	49·7	82·5	22·8	82·1	79·1	83·1
—	33·7	28·0	69·8	71·5	—	97·7	—	−1·7	57·9	84·9	—	—	—	—	—	—	—	—	—
—	—	—	70·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
11·9	34·0	28·0	70·0	70·9	70·9	98·8	98·8	−0·9	59·9	82·3	74·8	52·7	52·7	52·4	—	25·6	85·3	85·9	—
11·0	32·3	27·7	72·3	74·0	74·6	97·7	96·9	−1·7	59·7	85·7	78·0	43·3	42·5	41·9	—	—	—	—	—
11·2	33·4	26·9	77·7	74·9	75·1	103·8	103·4	2·9	60·2	80·5	63·4	62·5	61·7	61·8	—	25·3	80·0	82·1	—
—	33·7	26·1	73·8	78·9	78·9	93·6	93·6	−5·1	61·3	77·4	74·2	55·3	53·1	55·1	—	13·4	80·8	88·4	—
—	37·7	30·0	71·5	70·3	—	101·9	—	1·1	61·0	79·6	—	—	—	—	—	—	—	—	—
—	—	—	72·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	33·0	27·2	68·4	72·9	—	93·7	—	−4·6	59·8	82·4	—	—	—	—	—	—	—	—	—
—	32·0	28·8	75·4	73·9	73·0	102·0	103·3	1·5	60·7	90·0	70·4	50·7	50·7	50·6	—	29·5	75·0	76·9	82·5
—	35·3	—	78·2	71·8	71·2	109·9	110·0	6·5	62·1	—	—	58·7	57·0	57·4	—	30·3	93·0	90·5	—
—	36·1	26·8	76·6	74·5	73·3	102·8	104·5	2·1	61·1	74·2	67·4	50·9	51·2	50·2	—	39·4	74·7	76·6	84·3
—	33·2	—	73·3	73·0	—	100·4	—	0·3	60·0	—	80·9	35·3?	35·9?	35·5?	—	—	86·3	—	—
15·2	—	—	—	—	—	—	—	—	62·1	—	76·1	60·3	59·8	59·8	46·2	27·0	77·8	84·5	81·5
—	36·7	34·1	72·6	71·5	70·9	101·5	102·3	1·1	57·2	92·9	73·6	60·4	60·9	60·2	—	32·2	84·2	87·8	—
—	—	—	71·2	—	—	—	—	—	59·4	—	73·1	49·4	49·2	49·2	56·8	—	89·6	88·9	94·3
—	34·0	29·0	76·2?	72·3	72·6	105·3?	104·9?	3·9?	64·2	85·3	73·4	42·8	42·3	42·2	—	22·9	84·2	85·0	87·6
12·7	32·1	27·1	77·6	71·2	71·5	108·9	108·5	6·4	61·0	84·4	74·5	59·1	57·6	59·0	40·6	31·7	83·1	84·0	88·9
—	33·7	27·0	71·2	73·5	74·6	96·9	95·4	−2·3	61·5	80·1	—	—	—	—	—	—	83·0	—	—
—	—	—	68·1	—	—	—	—	—	64·0	—	77·6	57·9	56·9	57·3	—	27·4	83·4	83·8	—
—	36·1	29·0	70·4	69·3	—	101·6	—	1·1	61·2	80·3	—	—	—	—	—	—	—	—	—
—	33·5	27·3	76·7	74·3	74·9	103·2	102·4	2·4	61·7	81·5	72·7	47·7	48·0	47·6	—	47·6	85·5	84·7	—
—	—	—	73·3	—	—	—	—	—	55·4	—	62·2	59·0	57·4	59·0	—	27·7	79·4	82·5	83·5
—	33·9	26·9	70·3	75·4	74·9	93·3	94·0	−5·1	58·1	74·7	67·3	55·3	53·9	54·9	45·4	24·3	84·1	82·5	86·0
—	36·0	32·0	71·6	72·5	72·5	98·8	98·8	−0·8	62·7	88·9	69·6	53·6	53·0	53·6	—	37·0	83·2	84·9	—
13·7	34·8	27·7	71·7	75·5	76·1	94·9	94·2	−3·8	—	79·6	75·2	54·3	54·2	54·4	42·8	7·2	79·3	80·7	86·1
10·9	32·5	28·0	76·9	74·6	74·0	103·1	103·9	2·3	59·2	86·1	68·2	—	55·8	55·9	—	30·9	80·3	77·4	—
—	33·4	26·5	79·6	72·9	72·9	109·2	109·2	6·7	59·6	79·3	—	62·7	61·3	62·4	—	—	87·6	84·9	—
12·0	35·0	30·0	76·8	75·0	75·0	102·4	102·4	1·8	62·1	85·7	71·3	46·9	47·5	46·8	—	47·8	80·5	81·9	82·3
12·9	33·0	28·1	73·2	67·8	68·4	107·9	107·0	5·4	57·0	85·1	74·0	55·6	55·1	53·2	—	31·3	83·9	81·8	—

100 $\frac{O_2}{O_1}$, L	100 $\frac{O_2}{O_1''}$, R	100 $\frac{O_2}{Lacr.\,O_1}$, R	100 $\frac{G_2}{G_1'}$	100 $\frac{G_2}{G_1}$	100 $\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alv.* $P\angle$	*Pros.* $P\angle$	$\theta_1\angle$	$\theta_2\angle$	
83·7	—	—	—	—	—	66°·4	68°·6	44°·9	84°	82°	29°·6	15°·4	All sutures closed
83·5	86·9	89·5	—	—	—	62°·7	74°·0	43°·3	84°	82°·5	33°·3	10°·0	
82·4	86·0	87·6	71·2	67·2	—	70°·5	72°·9	36°·6	—	80°	—	—	Coronal and sagit
—	—	—	91·3	86·4	25·8	—	—	—	—	—	—	—	Coronal and sagit
72·8	81·5	85·1	77·5	71·0	37·8	73°·7	68°·0	38°·4	79°·5	75°·5	26° 8	11°·5	JR.
86·0	—	—	—	—	—	66°·4	72°·2	41°·4	85°	81°·5	28°·6	12°·8	Sagittal suture be
—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal and lamb
—	—	—	73·8	69·2	41·9	—	—	—	81°·5	80°·0	—	—	
78·9	—	—	88·4	—	32·8	62°·5	75°·1	42°·4	87°	84°	30°·5	11°·9	All sutures closed
—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures oblite
81·8	—	—	—	—	—	68°·8	73°·7	37°·7	84°	82°	27°·2	10°·3	Sagittal suture cl
—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture cl
83·4	89·9	89·9	86·8	78·5	28·2	65°·0	75°·0	40°·0	83°	80°·5	32°·0	8°·0	JR.
82·0	85·7	88·3	87·0	79·5	37·6	68°·6	70°·9	40°·5	80°·5	78°	30°·9	9°·6	JR.
78·0	—	—	—	—	—	71°·6	70°·2	38°·1	81°	78°	27°·4	10°·8	All sutures closed
78·8	—	—	—	—	—	65°·0	73°·0	42°·0	—	79°·5	—	—	
78·1	—	—	—	—	—	70°·0	69°·7	40°·3	80°	77°	30°·0	10°·3	Sagittal suture be
80·6	—	—	—	—	—	—	—	—	85°	—	—	—	Teeth worn. JR.
73·5	83·3	84·0	85·5	76·1	30·4	69°·6	71°·0	39°·4	81°·5	79°	28°·9	10°·5	J=.
82·9	82·1	—	—	—	—	—	—	—	86°	82°·5	—	—	Coronal and sagit
76·7	85·1	85·1	88·1	79·3	31·2	60°·0	82°·7	37°·2	91°	89°	29°·0	8°·3	JR.
82·7 ?	—	—	—	—	—	—	—	—	—	—	—	—	Teeth worn.
83·3	87·4	87·9	92·7	—	34·0	—	—	—	88°	86°	—	—	Coronal suture cl
80·5	—	—	83·1	77·9	33·9	72°·7	69°·0	38°·3	81°·5	80°	25°·8	12°·5	JR.
74·7	—	—	92·4	79·8	34·2	68°·7	72°·0	39°·3	82°·5	81°	28°·8	10°·5	Sagittal and lam
76·5	—	—	—	—	—	72°·3	72°·1	35°·6	—	75°	—	—	Coronal suture b
79·1	83·1	87·0	88·3	80·7	—	67°·0	69°·0	44°·0	82°·5	81°	30°·5	13°·5	Coronal suture b
—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture p
—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagi
85·9	—	—	80·6	75·8	32·2	71°·9	68°·0	40°·1	82°	78°	26°·1	14°·0	Coronal and sagi
—	—	—	—	—	30·2	64°·7	72°·9	42°·4	84°	81°	31°·3	11°·1	JR.
82·1	—	—	91·0	84·5	26·9	66°·8	74°·6	38°·6	92°	90°·5	21°·2	17°·4	Coronal and sagi
88·4	—	—	—	—	—	70°·2	70°·0	39°·7	89°	87°	20°·8	19°·0	Three teeth lost
—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagi
—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagi
—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture o
76·9	82·5	84·9	—	—	—	69°·5	73°·1	37°·4	80°·5	79°	30°·0	7°·4	Six teeth lost be
90·5	—	—	—	—	—	—	—	—	—	—	—	—	All sutures close
76·6	84·3	84·3	76·9	69·8	—	67°·0	74°·6	38°·5	94°	—	19°·0	19°·4	JR.
—	—	—	—	—	—	68°·7	69°·3	42°·0	—	74°	—	—	Coronal suture b
84·5	81·5	85·3	—	—	45·8	—	—	—	—	—	—	—	Sagittal suture o
87·8	—	—	—	—	—	68°·0	71°·0	41°·0	—	—	—	—	Teeth worn and
88·9	94·3	—	—	—	—	—	—	—	—	—	—	—	All sutures close
85·0	87·6	91·9	—	—	—	69°·8	68°·0	42°·2	89°	87°	21°·2	21°·0	Coronal and lam
84·0	88·9	91·7	75·8	70·9	34·3	72°·9	67°·7	39°·5	88°·5	87°	18°·6	20°·8	Sagittal suture b
—	—	—	—	—	—	—	—	—	—	—	—	—	J=.
83·8	—	—	80·8	70·8	—	—	—	—	90°	88°	—	—	Sagittal suture b
—	—	—	—	—	—	—	—	—	—	—	—	—	JR.
84·7	—	—	—	—	—	65°·0	73°·8	41°·2	—	87°	—	—	Sagittal suture b
82·5	83·5	85·8	85·5	80·1	—	—	—	—	—	—	—	—	
82·5	86·0	90·2	—	—	—	68°·2	73°·9	37°·8	87°·5	86°	24°·3	13°·6	Sagittal suture p
84·9	—	—	—	—	—	62°·0	78°·0	40°·0	83°	81°	35°·0	5°·0	All sutures close
80·7	86·1	91·2	77·8	72·5	38·7	70°·7	67°·6	41°·7	87°	86°	22°·3	19°·4	JL.
77·4	—	—	85·5	77·0	31·9	66°·9	73°·6	39°·5	92°	91°	21°·1	18°·4	JR.
84·9	—	—	—	—	—	—	—	—	—	—	—	—	JR.
81·9	82·3	—	83·1	79·4	33·5	64°·8	76°·8	38°·4	—	81°	—	—	JR.
81·8	—	—	77·4	71·4	36·1	67°·7	71°·9	40°·2	—	—	—	—	Coronal suture b

Remarks
All sutures closed and partly obliterated. 3 molars lost before death. JR.
Coronal and sagittal sutures beginning to close. Teeth markedly worn and 1 incisor lost before death. JR.
Coronal and sagittal sutures beginning to close. Teeth worn.
JR.
Sagittal suture beginning to close. JR.
Sagittal and lambdoid sutures partly obliterated, coronal closed. Teeth worn.
All sutures closed and nearly obliterated. Teeth markedly worn, 2 molars lost before death. JL.
All sutures obliterated. JR.
Sagittal suture closed. Teeth worn, but apparently none lost before death. J=.
Sagittal suture closed.
JR.
JR.
All sutures closed, coronal partly obliterated. JR.
Sagittal suture beginning to close. 1 molar lost before death. JL.
Teeth worn. JR.
J=.
Coronal and sagittal sutures closed and partly obliterated.
JR.
Teeth worn.
Coronal suture closed and partly obliterated, sagittal closing.
JR.
Sagittal and lambdoid sutures nearly obliterated, coronal closed. JL.
Coronal suture beginning to close. JR.
Coronal suture beginning to close. JR.
Sagittal suture partly obliterated, coronal closed. Teeth worn.
Coronal and sagittal sutures obliterated, lambdoid closing.
Coronal and sagittal sutures closing. Teeth worn. JL.
JR.
Coronal and sagittal sutures closing. JR.
Three teeth lost before death. JR.
Coronal and sagittal sutures closing. JR.
Coronal and sagittal sutures closing.
Sagittal suture obliterated, lambdoid partly obliterated, coronal closing. JR.
Six teeth lost before death. JR.
All sutures closed. Teeth worn and 4 (?) lost before death.
JR.
Coronal suture beginning to close. Teeth worn.
Sagittal suture obliterated, coronal and lambdoid closing.
Teeth worn and 2 lost before death. JR.
All sutures closed. Teeth worn.
Coronal and lambdoid sutures closed, sagittal closing. Teeth worn. JR.
Sagittal suture beginning to close. JL.
J=.
Sagittal suture beginning to close.
JR.
Sagittal suture beginning to close. Teeth worn and 8 or more lost before death. J=.
Sagittal suture partly obliterated, lambdoid closing. JR.
All sutures closed. 6 teeth lost before death. JL.
JL.
JR.
JR.
JR.
Coronal suture beginning to close. JR.

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Se N
2
2
2

Serial No.	Grave Mark	Probable Dynasty	C	B′	B″	B	Biasterionic B	F	L	H′	H	OH	LB
208	1045 D	XIII	—	87·9	—	124	103·0	178	180	128·5	—	—	101·
210	437 A	XII	1217	88·2	111	130	100·3	181	182	128	128·5	108	104·
211	1045 H	XIII	1266	92·6	—	131	99·1	183·5	183	135	—	—	102·
212	2000 F	,,	1239	90·8	108·5	128	102·3	180·5	180	135	134	109	102·
213	1805 H	,,	—	82·0	104	124·5	102·0	171	171	129	131	108	99·
214	1096 D	,,	—	86·0	104	121	94·6	173	172·5	125	125·5	107	90·
215	3401 M	,,	—	91·2	108·5	135	—	171·5	171	129	128	111	93·
216	1045 G	,,	1229	92·3	—	133	104	178	179	127	127	102	99·
217	1058 A	,,	—	90·2	—	126·5	104·2	182·5	182·5	124	—	—	101·
218	1082 KX	,,	1350	87·8	114·5	130	102·2	177	178·5	127	127	108·5	93·
219	1805 JA	,,	—	91·5	—	139	102·2	179·5	178·5	126	—	112	92·
220	III 23	XII	—	86·3	—	133·5	105·2	176·5	179	—	—	111	—
222	1805 BA	XIII	—	86·0	103	126	99·3	164·5	166	—	—	104	—
223	1045	,,	—	88·5	—	132	93·2	180·5	181	—	—	113	—
224	—	—	—	97·2	120·5	136·5	105·0	180	180	136?	—	111·5	—
225	1603 A	XIII	1349	87·6	104·5	131	100·0	182·5	182	134	134	109	94·
226	319 K	XII	1278	87·1	104·5	138	97·1	177	176·5	133	134	108	94·
227	1091	XIII	1020	87·0	102	122·5	98·0	167	168	120	122	103	90·
228	1026 E	,,	1211	93·4	—	133	102·7	173	174	122	123	106	96·
229	1600 XC	,,	—	89·0	—	124	101·2	180·5	179·5	128	127	106	92·
230	1803 C	,,	—	87·0	104·5	131	94·2	182	180·5	133	133	108	94·
231	1049 X	,,	1148	89·2	102	123·5	100·0	171	171	127	126·5	107	96·
232	XB/TB	,,	—	89·2	—	137	106·7	180	182	126	—	—	95·
233	XB (UC)	,,	—	90·0	—	127·5	108·0	174	176	128·5	131·5	110	99·
234	XB/OB	,,	—	86·8	107	131	102·2	178·5	180	128·5	130·5	112	94·
235	XB/UC	,,	1450	92·0	—	137	103·4	188·5	190	137·5?	139·5	118	105·
236	XB (A)/ZB	,,	1173	90·7	—	132	102·1	176·5	176	—	119·5	103	93·
237	XB/ZB	,,	1232	86·0	103	127·5	104·0	173·5	175·5	125	126	106	93·
238	XB/ZC (C)	,,	1221	89·8	—	129·5	99·0	183·5	183	—	128	106	101·
239	IV B/YA	XII	—	92·0	—	126	—	182	183	—	—	—	—
240	III 301	,,	1246	93·0	—	133	99·2	177	177	133·5	132·5	111·5	100·
242	IV B/OA	,,	—	89·2	—	126	—	178·5	179·5	—	—	—	—
243	III 301/U	,,	—	84·2	98	123	96·1?	181	181	—	—	106	—
244	III B/EA	,,	—	88·0	101	122·5	101·0	183	184	—	—	—	—
245	IV 414 E	,,	—	94·0	112	129·5	100·2	185	184·5	—	—	—	—
246	15 A/B	,,	—	92·8	—	134	105·8	189	188·5	132·5	133	108	98·
248	1356	XIII	—	—	—	129·5	97·0	172	175	—	—	—	—
249	1805	,,	—	88·0	106	126	105·0	178·5	179	138	138	110·5	100·
251	445 (ii)	XII	1102	84·0	103	128	97·7	169	170	126·5	127	103	93·
252	449 A	,,	1278	89·3	114·5	130	105·0	177	176·5	129·5	129·5	113	94·
253	1073	XIII	1270	90·8	—	133	100·2	178·5	179	129·5	131	106·5	101·
255	1002 (i)	,,	1302	86·7	109	137	102·0	171	172	127·5	129·5	111	92·
257	450 B	XII	1265	87·0	106	132	102·8	178	178·5	134·5	136	111·5	101·
258	1627	XIII	1099	92·5	103	122	95·3	168	169	120	120·5	104	96·
259	1619 (i)	,,	1179	98·0	—	131	94·4	175	176	130	130·5	108	103·
261	361 (ii)	XII	—	85·8	—	128·5	102·7	177·5	178·5	127·5	127	110	95·
263	2000 J	XIII	—	86·8	—	127	99·9	169·5	170	—	135	108	101·
264	2311 G	,,	—	81·4	—	138	—	176	177	—	—	—	—
266	1058 D	,,	—	90·5	116	138	102·9	187	187·5	128	—	—	98·
267	1805/NA	,,	1248	93·1	113·5	131	105·0	181·5	180	127·5	127·5	104·5	93·
268	1044 Y	,,	—	89·4	—	130	99·3	180	180	128	129	112	95·
269	K 425 A	XII	1216	88·1	—	135	102·8	175	173·5	133	132·5	107·5	95·
270	K 307 B	,,	—	86·4	104·5	130	105·2	183	181·5	122	123·5	108	94·
271	444 M	,,	1136	86·4	—	130·5	101·1	173	175	124	127	106	101·
272	—	—	—	—	—	135	—	182·5	182	125·5	127	105	100·
274	1000 B	XIII	1261	88·7	107	130	104·0	189	189·5	130	130·5	110	99·
275	1091 A	,,	—	92·8	110	131	104·0	184	185	129	130	104·5	103·

H	*OH*	*LB*	S_1'	S_2'	S_3'	S_1	S_2	S_3	*S*	*U*	*Q'*	*G'H*	*GL*	*GB*	*J*	*NH'*	*NH, R*	*NH, L*	*NB*
—	—	101·3	109·0	108·7	92·0	125	118	115	358	480	—	—	—	—	—	—	—	—	—
128·5	108	104·7	110·2	109·2	93·7	127	123	112	362	487	291	66·0	97·1	89·7	112	47·0	47·4	47·2	24·0
—	—	102·0	109·6	117·7	96·5	126	131	116	373	500	—	—	—	—	—	44·0	—	—	—
134	109	102·0	115·4	113·1	90·3	134	128	105	367	494	295	70·5	95·5	94·3	123	51·2	52·4	51·3	25·7
131	108	99·1	100·2	106·8	89·5	112	120	106	338	497	282	62·8	101·4	93·5	—	47·2	46·9	47·3	26·3
125·5	107	90·0	107·9	114·4	91·0	124	130	109	363	471	285	—	—	82·5	111	48·8	49·1	49·2	24·7
128	111	93·4	113·8	103·2	89·0	133	116	108	357	480	304	—	—	90·4	—	42·7	43·2	43·2	24·2
127	102	99·9	108·4	108·7	92·1	122	120	109	351	498	286	65·4	94·2	97·6	125	46·1	46·1	46·3	27·6
—	—	101·0	104·4	107·9	96·6	120	120	122	362	495	—	—	—	—	—	—	—	—	—
127	108·5	93·0	112·7	115·6	90·3	133	129	103	365	495	295	—	—	92·7	125	45·0	44·9	45·1	27·8
—	112	92·0	109·7	118·3	91·3	130	135	109	374	483	303	61·8	87·8	—	—	44·1	—	—	—
—	111	—	106·7	109·6	102·2	119	123	119	361	497	306	64·7	—	87·0	—	49·4	49·2	49·7	29·2
—	104	—	102·3	94·3	97·5	116	104	115	335	465	282	62·4	—	96·7	—	47·8	46·1	47·0	26·1
—	113	—	112·3	116·0	95·6	127	129	113	369	491	304	70·8	100·4	93·5	—	48·2	48·6	48·6	26·9
—	111·5	—	113·7	115·0	—	134	130	—	—	511	305	64·9	87·1	90·1	—	49·8	50·2	49·4	25·7
134	109	94·0	111·0	115·1	96·0	131	125	119	375	496	292	62·9	90·5	83·4	115·5	47·8	47·4	48·0	26·0
134	108	94·7	106·8	124·8	88·0	123	145	102	370	480	295	—	—	90·2	—	45·0	43·8	45·3	26·4
122	103	90·0	99·1	110·7	87·8	113	124	102	339	464	284	67·0	87·2	96·3	117	45·3	45·5	45·5	23·4
123	106	96·1	102·1	118·3	88·8	114	132	102	348	492	296	70·0	100·3	91·0	120	49·8	49·8	49·9	24·0
127	106	92·1	112·0	103·6	106·0	133	114	129	376	495	288	71·4	88·9	88·9	115·5	50·7	51·3	51·1	24·4
133	108	94·0	108·3	112·8	98·1	126	124	122	372	492	288	65·5	96·6	—	—	45·0	45·3	45·1	23·3
126·5	107	96·1	106·6	111·0	88·0	122	127	103	352	472	290	—	—	—	—	51·6	—	—	—
—	—	95·0	111·2	117·0	96·1	127	130	115	372	505	—	—	—	—	—	—	—	—	—
131·5	110	99·1	101·6	110·5	96·0	111	125	117	353	487	296	67·9	97·8	83·8	121	47·2	47·5	47·3	27·8
130·5	112	94·1	108·0	112·8	103·2	124	125	123	372	496	296	67·7	92·2	97·1	120·5	48·0	48·0	47·7	25·8
139·5	118	105·0	117·8	119·7	100·2	133	134	120	387	516	321	—	—	—	—	—	—	—	—
119·5	103	93·0	105·5	112·0	88·4	121	127	112	360	491	288	66·2	93·6	96·2	—	48·8	50·0	49·3	25·8
126	106	93·6	104·6	111·5	89·8	118	127	108	353	486	286	64·4	87·0	88·2	—	47·7	47·5	48·1	23·5
128	106	101·3	—	—	—	—	—	—	360	498	290	67·5	91·5	91·9	119	50·0	49·9	50·8	26·9
—	—	—	106·7	116·5	100·5	120	130	122	372	—	—	—	—	—	—	—	—	—	—
132·5	111·5	100·7	107·4	110·2	99·2	126	122	118	366	487	308	64·4	91·1	90·0	—	45·9	45·5	46·0	21·7
—	—	—	—	—	—	—	—	—	360	487?	—	—	—	—	—	—	—	—	—
—	106	—	109·6	112·0	94·7	124	128	114	366	489	284	—	—	—	—	—	—	—	—
—	—	—	113·5	114·4	95·5	129	127	111	367	493	—	—	—	—	—	—	—	—	—
—	—	—	109·8	114·2	93·8	129	130	110	369	510	—	—	—	—	—	—	—	—	22·4
133	108	98·0	114·1	115·5	99·8	132	127	126	385	513?	—	68·5	91·0	89·2	—	—	—	—	26·0
—	—	—	108·9	112·5	—	122	125	—	—	484?	—	—	—	95·7	122	50·4	50·0	51·0	28·0
138	110·5	100·0	111·6	107·4	105·4	127	120	124	372	492	293	67·6	—	—	—	—	—	—	—
127	103	93·0	99·4	105·8	94·5	116	118	112	345	472	287	70·0	87·9	92·0	121	53·7	54·0	54·0	31·4
129·5	113	94·0	111·5	111·8	92·0	132	125	110	366	490	281	65·7	90·7	90·3	118	48·8	49·7	49·0	26·3
131	106·5	101·0	107·4	—	—	123	—	—	361	492	294	65·0	90·3	92·8	119	44·8	44·4	45·0	23·3
129·5	111	92·2	110·3	113·2	91·6	124	129	109	362	482	299	62·8	97·5	94·9	122	45·9	45·7	46·0	24·7
136	111·5	101·2	107·4	114·4	98·7	120	130	114	365	495	300	67·0	—	85·7	126	44·6	45·0	45·0	26·0
120·5	104	96·9	103·9	105·0	88·4	117	120	102	338	472	280	59·2	—	92·1	116·5	49·3	49·4	50·0	23·4
130·5	108	103·3	111·9	102·0	89·5	130	114	108	352	489	295	67·5	103·0	—	—	44·0	45·0	44·6	24·5
127	110	95·0	104·4	119·0	91·4	117	135	110	362	480	290	65·2	—	92·2	—	46·2	47·7	47·8	25·3
135	108	101·0	—	—	—	—	—	—	347	469	296	71·4	97·7	94·9	—	46·4	46·0	47·2	26·3
—	—	—	110·5	111·3	—	124	128	—	—	500	—	—	—	—	—	48·3	48·8	48·4	25·2
—	—	98·9	—	—	—	—	—	—	371	519?	—	—	—	—	—	—	—	—	—
127·5	104·5	93·0	109·0	117·1	95·1	130	128	111	369	499	285	65·3	94·4	101·0	125	47·4	48·0	47·3	25·4
129	112	95·2	109·5	119·8	89·8	126	136	107	369	495	303	—	—	—	—	—	—	—	—
132·5	107·5	95·4	110·5	114·7	93·1	130	130	107	366	491	291	67·8	97·0	98·9	124	45·1	45·2	45·3	27·0
123·5	108	94·0	106·0	113·2	95·0	122	125	120	367	497	292	60·5	95·0	83·0	—	45·7	45·4	45·9	25·7
127	106	101·2	102·7	108·2	96·0	115	120	112	347	487	295	66·9	99·8	93·0	124·5	49·2	49·3	49·9	24·3
127	105	100·0	104·0	—	—	119	—	—	356	—	291	63·2	—	87·5	—	51·0	—	51·1	28·3
130·5	110	99·1	110·0	117·9	99·8	123	131	120	374	504	287	68·5	97·8	97·2	121	49·5	50·0	50·7	23·0
130	104·5	103·1	111·0	118·0	94·8	127	130	113	369	506	301	—	—	89·7	—	54·3	55·6	54·3	—

TABLE IX. *Individual Measurements of Femo*

?	*NH, L*	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	*O_1R*	*O_1L*	*$O_1'R$*	*Lacr. $O_1'R$*	*O_2R*	*O_2L*	*G_1'*	*G_1*	*G_2*	*EH*	*fml*
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	47·2	24·0	—	—	—	4·5	11·4	39·0	37·8	37·4	36·9	32·0	31·0	—	—	34·8	10·9	32·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	42·8	—	35·8
	51·3	25·7	—	—	—	2·1	9·6	40·5	40·6	38·0	—	31·1	31·9	48·0	51·3	37·1	11·6	35·6
	47·3	26·3	9·8	21·1	30·5	3·0	9·1	39·2	40·0	38·1	36·2	31·1	31·9	—	—	38·2	11·2	37·2
	49·2	24·7	10·4	18·9	31·5	3·5	11·0	39·2	39·1	38·7	37·3	31·8	32·6	42·3	45·8	33·7	13·0	34·2
	43·2	24·2	9·2	20·2	30·0	3·0	8·0	39·0	39·2	37·7	34·4	29·0	29·0	—	—	36·4	11·1	31·2
	46·3	27·6	—	—	—	2·0	11·8	39·6	39·1	—	—	30·8	32·3	—	—	36·3	9·8	33·9
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35·3
	45·1	27·8	—	—	—	—	—	38·4	38·3	36·1	—	31·8	33·0	—	—	42·9	—	35·5
	—	—	—	—	—	—	—	40·2	—	—	—	32·0	—	—	—	—	—	38·6
	49·7	29·2 ?	—	—	—	—	—	39·6	38·7	—	—	33·0	33·0	—	—	—	—	—
	47·0	26·1	12·5	22·8	35·5	4·0	12·2	37·9	38·0	36·0	35·6	30·8	31·0	45·4	50·0	39·3	11·1	—
	48·6	26·9	—	—	—	—	—	39·2	40·3	—	—	34·3	33·4	47·3	51·1	37·1	14·3	35·3
	49·4	25·7	9·7	23·7	32·0	2·9	11·0	41·1	40·0	38·9	37·5	35·9	35·0	41·9	46·1	—	—	—
	48·0	26·0	—	—	—	2·8	8·0	38·6	40·0	—	—	34·0	32·8	—	—	37·0	—	36·4
	45·3	26·4	—	—	—	—	—	38·4	—	—	—	29·0	—	—	—	40·0	9·1	33·3
	45·5	23·4	10·1	23·6	30·5	2·5	8·0	37·3	38·3	37·0	35·8	31·0	30·9	41·8	45·2	41·0	13·0	33·1
	49·9	24·0	—	—	—	3·5	8·2	38·9	39·0	—	—	34·1	36·0	48·9	53·3	36·1	12·2	33·1
	51·1	24·4	—	—	—	—	—	39·0	39·7	—	—	35·8	36·8	44·9	48·0	38·7	13·6	36·1
	45·1	23·3	9·8	21·1	32·5	1·7	7·1	36·3	37·9	34·2	33·4	31·0	30·8	49·0	51·8	33·2	9·1	34·2
	—	—	—	—	—	—	—	—	39·3	—	—	—	31·9	—	—	—	—	31·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35·5
	47·3	27·8	10·0	21·3	32·5	3·7	12·9	39·7	41·5	37·3	—	33·0	32·0	48·9	54·0	36·2	15·2	35·6
	47·7	25·8	—	—	—	2·3	10·0	37·2	37·4	35·7	34·4	32·3	32·2	46·7	48·8	—	—	33·2
	—	—	—	—	—	—	—	—	42·0	—	—	—	35·0	—	—	—	—	36·0
	49·3	25·8	—	—	—	0·5	4·0	40·0	41·2	38·9	37·2	37·0	37·0	46·0	50·0	—	—	33·2
	48·1	23·5	—	—	—	2·8 ?	9·0 ?	38·5	38·0	—	36·0	34·2	34·0	43·1	46·2	—	—	39·9
	50·8	26·9	—	—	—	4·7	14·4	39·2	38·1	38·0	37·8	31·3	30·0	49·2	54·1	35·0	14·2	35·6
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	46·0	21·7	—	—	—	4·3	11·3	40·5	39·1	—	—	30·1	30·6	44·2	48·1	34·7	9·7	32·1
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	33·8
	—	—	—	—	—	—	—	—	—	—	—	—	31·7	—	—	—	—	—
	—	22·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	26·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	38·4
	51·0	28·0	—	—	—	2·3	10·0	41·4	40·1	39·2	37·0	30·6	31·7	45·1	48·2	42·0	13·0	33·6
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	54·0	31·4	—	—	—	4·1	10·3	40·0	40·4	37·2	36·9	30·8	30·8	47·2	53·1	40·3	—	33·6
	49·0	26·3	—	—	—	5·1	11·8	39·7	38·8	—	—	33·2	33·9	45·0	49·1	—	—	34·8
	45·0	23·3	8·0	20·8	29·0	2·2	8·9	40·0	40·3	39·2	36·7	32·0	33·0	45·0	47·9	39·6	16·4	35·1
	46·0	24·7	10·9	20·0	34·0	4·3	14·0	40·0	41·0	39·3	38·3	29·4	29·3	44·9	47·6	—	—	34·5
	45·0	26·0	—	—	—	4·0	10·0	40·0	40·0	—	—	29·6	30·7	46·4	52·6	39·2	11·9	32·2
	50·0	23·4	—	—	—	2·2	7·7	40·9	39·2	38·0	37·2	33·8	32·8	—	—	37·3	9·2	34·8
	44·6	24·5	—	27·6	—	—	13·3	41·3	40·4	38·6	37·9	28·7	28·8	—	—	36·0	9·0	31·8
	47·8	25·3	12·2	24·6	38·5	3·8	12·9	38·4	38·7	36·0	35·9	34·0	34·2	54·0	55·0	40·0	13·4	33·0
	47·2	26·3	—	—	—	4·2	10·2	39·0	39·0	37·0	36·2	33·2	32·2	51·2	52·6	37·3	9·7	35·0
	48·4	25·2	11·4	20·0	34·5	4·9	13·1	40·3	39·2	39·1	37·8	32·2	33·0	49·0	53·1	39·3	17·0	35·0
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	38·2
	47·3	25·4	11·5	22·1	32·5	2·2	7·4	43·8	43·0	40·3	38·3	35·0	35·0	—	—	41·2	—	39·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34·0
	45·3	27·0	—	—	—	2·2	8·3	43·4	41·2	—	—	32·3	31·9	48·1	52·0	37·6	13·1	33·0
	45·9	25·7	—	—	—	1·6	8·1	40·4	41·0	—	—	30·3	31·0	47·0	49·8	—	—	34·6
	49·9	24·3	—	—	—	3·7	13·2	40·7	38·9	37·9	—	31·0	32·0	48·2	54·8	42·0	12·6	30·7
	51·1	28·3	—	—	—	—	—	—	40·0	—	—	—	31·9	—	—	37·9	—	39·0
	50·7	23·0	8·9	22·8	29·0	—	9·1	40·0	39·0	39·4	38·0	33·1	34·0	47·0	48·4	39·8	11·8	35·0
	54·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	33·3

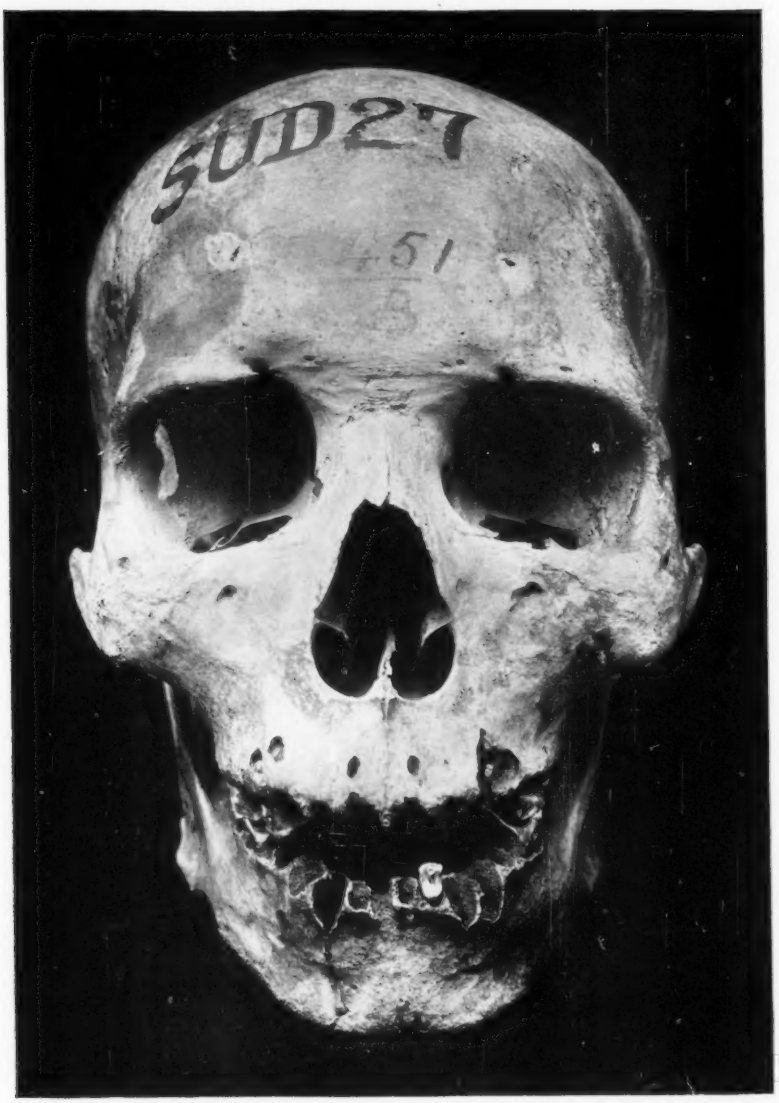
...urements of Female Kerma Skulls (continued).

G_2	EH	fml	fmb	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{H}{L}$	100 $\frac{B}{H'}$	100 $\frac{B}{H}$	100 $\frac{(B-H')}{L}$	$Oc.\ I.$	100 $\frac{fmb}{fml}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH}, R$	100 $\frac{NB}{NH}, L$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{O_1}, L$
—	—	—	27·3	68·9	71·4	—	96·5	—	−2·5	57·1	—	—	—	—	—	—	—	—	—
34·8	10·9	32·7	27·9	71·4	70·3	70·6	101·6	101·2	1·1	60·4	83·5	73·6	51·1	50·6	50·8	—	39·5	82·0	82·0
42·8	—	35·8	26·0	71·6	73·8	—	97·0	—	−2·2	59·9	72·6	—	—	—	—	—	—	—	—
37·1	11·6	35·6	26·9	71·1	75·0	74·4	94·8	95·5	−3·9	63·4	75·6	74·8	50·2	49·0	50·0	—	21·9	76·6	78·6
38·2	11·2	37·2	26·9	72·8	75·4	76·6	96·5	95·0	−2·6	61·3	76·4	67·2	55·7	56·1	55·6	46·5	33·0	77·7	79·8
33·7	13·0	34·2	27·5	70·1	72·5	72·7	96·8	96·4	−2·3	60·2	80·4	—	50·6	50·3	50·2	55·0	31·8	81·1	83·4
36·4	11·1	31·2	25·3	78·9	75·4	74·9	104·7	105·5	3·5	59·1	81·1	—	56·7	56·0	56·0	45·5	37·5	74·4	74·0
36·3	9·8	33·9	31·0	74·3	70·9	70·9	104·7	104·7	3·3	61·4	91·4	67·0	59·9	59·9	59·6	—	16·9	77·7	82·6
—	—	35·3	28·0	69·3	67·9	—	102·0	—	1·4	56·5	79·3	—	—	—	—	—	—	—	—
42·9	—	35·5	29·3	72·8	71·1	71·1	102·4	102·4	1·7	66·3	82·5	—	61·8	61·9	61·6	—	—	82·8	86·2
—	—	38·6	—	77·9	70·6	—	110·3	—	7·3	60·5	—	—	—	—	—	—	—	79·6	—
—	—	—	—	74·6	—	—	—	—	—	63·3	—	74·4	59·1 ?	59·3 ?	58·7 ?	—	—	83·3	82·6
39·3	11·1	—	—	75·9	—	—	—	—	—	61·8	—	64·5	54·6	56·6	55·5	54·8	32·8	81·3	81·6
37·1	14·3	35·3	—	72·9	—	—	—	—	—	61·5	—	75·7	55·8	55·3	55·3	—	—	87·5	82·9
—	—	—	—	75·8	75·6 ?	—	100·3 ?	—	0·3 ?	—	—	72·0	51·6	49·4	52·0	40·9	26·4	87·3	87·5
37·0	—	36·4	—	72·0	73·6	73·6	97·8	97·8	−1·6	57·5	—	75·4	54·4	54·8	54·2	—	35·0	88·1	82·0
40·0	9·1	33·3	27·9	78·2	75·4	75·9	103·8	103·0	2·8	63·9	83·8	—	58·7	60·3	58·3	—	—	75·5	—
41·0	13·0	33·1	28·8	72·9	71·4	72·6	102·1	100·4	1·5	63·5	87·0	69·6	51·7	51·4	51·4	42·8	31·2	83·1	80·7
36·1	12·2	33·1	26·8	76·4	70·1	70·7	109·0	108·1	6·3	65·3	81·0	76·9	48·2	48·2	48·1	—	42·7	87·7	93·2
38·7	13·6	36·1	27·0	69·1	71·3	70·7	96·9	97·6	−2·2	58·8	74·8	80·3	48·1	47·6	47·7	—	—	91·8	92·7
33·2	9·1	34·2	27·2	72·6	73·7	73·7	98·5	98·5	−1·1	57·3	79·5	—	51·8	51·4	51·7	46·6	23·9	85·4	81·3
—	—	31·5	26·3	72·2	74·3	74·0	97·2	97·6	−2·0	62·7	83·5	—	—	—	—	—	—	—	81·2
—	—	35·5	28·6	75·3	69·2	—	108·7	—	6·0	60·2	80·6	—	—	—	—	—	—	—	—
36·2	15·2	35·6	29·0	72·4	73·0	74·7	99·2	97·0	−0·9	58·7	81·5	81·0	58·9	58·5	58·8	46·9	28·7	83·1	77·1
—	—	33·2	25·9	72·8	71·4	72·5	101·9	100·4	1·4	60·6	78·0	69·7	53·7	53·7	54·1	—	23·0	86·8	86·1
—	—	36·0	33·3	72·1	72·4 ?	73·4	99·6 ?	98·2	−0·3 ?	60·2	92·5	—	—	—	—	—	—	—	83·3
—	—	33·2	27·0	75·0	—	67·9	—	110·5	—	56·3	81·3	68·8	52·9	51·6	52·3	—	12·5	92·5	89·8
—	—	39·9	29·2	72·6	71·2	71·8	102·0	101·2	1·4	59·8	73·2	73·0	49·3	49·5	48·9	—	31·1 ?	87·9	89·5
35·0	14·2	35·6	31·0	70·7	—	70·0	—	101·2	—	—	87·1	73·4	53·8	53·9	52·9	—	32·6	79·8	78·7
—	—	—	—	68·9	—	—	—	—	—	59·0	—	—	—	—	—	—	—	—	—
34·7	9·7	32·1	26·3	75·1	75·4	74·9	99·6	100·4	−0·3	60·8	81·9	71·5	47·3	47·7	47·2	—	38·1	74·3	78·3
—	—	33·8	—	70·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	68·0	—	—	—	—	—	59·7	—	—	—	—	—	—	—	—	—
—	—	—	—	66·6	—	—	—	—	—	63·5	—	—	—	—	—	—	—	—	—
—	—	38·4	—	70·2	—	—	—	—	—	62·3	—	—	—	—	—	—	—	—	—
42·0	13·0	33·6	27·3	71·1	70·3	70·5	101·1	100·7	0·8	56·5	81·2	71·6	55·5	56·0	54·9	—	23·0	73·9	79·0
—	—	—	—	74·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
40·3	—	33·6	29·0	70·4	77·1	77·1	91·3	91·3	−6·7	62·1	86·4	73·5	58·5	58·1	58·1	—	39·8	77·0	76·2
—	—	34·8	29·7	75·3	74·5	74·7	101·2	100·8	0·9	61·2	85·3	77·5	53·9	52·9	53·7	—	43·2	83·6	87·4
39·6	16·4	35·1	27·0	73·7	73·4	73·4	100·4	100·4	0·3	60·3	76·9	77·5	52·0	52·4	51·8	38·5	24·7	80·0	81·9
—	—	34·5	29·2	74·3	72·3	73·2	102·7	101·5	2·0	—	84·6	70·0	53·8	54·0	53·7	54·5	30·7	73·5	71·5
39·2	11·9	32·2	27·6	79·7	74·1	75·3	107·5	105·8	5·5	60·8	85·7	66·2	58·2	57·8	57·8	—	40·0	74·0	76·7
37·3	9·2	34·8	28·3	73·9	75·3	76·2	98·1	97·1	−1·4	64·4	81·3	74·7	47·5	47·4	46·8	—	28·6	82·6	83·7
36·0	9·0	31·8	25·0	72·2	71·0	71·3	101·7	101·2	1·2	64·5	78·6	64·3	55·7	54·4	54·9	—	—	69·5	71·3
40·0	13·4	33·0	26·3	74·4	73·9	74·1	100·8	100·4	0·6	59·5	79·7	—	54·8	53·0	52·9	49·6	29·5	88·5	88·4
37·3	9·7	35·0	25·0	72·0	71·4	71·1	100·8	101·2	0·6	59·8	71·4	70·7	56·7	57·2	55·7	—	41·2	85·1	82·6
39·3	17·0	35·0	33·0	74·7	—	73·2	—	94·1	—	—	94·3	75·2	52·2	51·6	52·1	57·0	37·4	79·9	84·2
—	—	—	—	78·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	38·2	28·0	73·6	68·3	—	107·8	—	5·3	—	73·3	—	—	—	—	—	—	—	—
41·2	—	39·7	28·1	72·8	70·8	70·8	102·7	102·7	1·9	63·0	70·8	64·7	53·6	52·9	53·7	52·0	29·7	79·9	81·4
—	—	34·0	27·1	72·2	71·1	71·7	101·6	100·8	1·1	60·7	79·7	—	—	—	—	—	—	—	—
37·6	13·1	33·0	28·0	77·8	76·6	76·4	101·5	101·9	1·1	65·1	84·8	68·5	59·9	59·7	59·6	—	26·5	74·4	77·4
—	—	34·6	—	71·6	67·2	68·0	106·6	105·3	4·4	56·5	—	73·1	56·2	56·5	55·7	—	19·8	75·0	75·6
42·0	12·6	30·7	30·4	74·6	70·9	72·6	105·2	102·8	3·7	63·0	99·0	71·9	49·4	49·3	48·7	—	28·0	76·2	82·3
37·9	—	39·0	34·0	74·2	68·9	69·8	107·5	106·3	5·2	—	87·1	72·2	55·5	—	55·4	—	—	—	79·7
39·8	11·8	35·0	33·1	68·6	68·6	68·9	100·0	99·6	0·0	59·9	94·6	70·5	46·5	46·0	45·4	39·0	—	82·7	79·7
—	—	33·3	26·2	70·8	69·7	70·3	101·5	100·8	1·1	60·6	78·7	—	—	—	—	—	—	—	—

$100\frac{O_2}{O_1}, R$	$100\frac{O_2}{O_1}, L$	$100\frac{O_2}{O_1}, R$	$100\frac{O_2}{Lacr.\ O_1}, R$	$100\frac{G_2}{G_1}$	$100\frac{G_2}{G_1}$	$100\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alv.* $P\angle$	*Pros.* $P\angle$	$\theta_1\angle$	$\theta_2\angle$	
—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures closed, corona
82·0	82·0	85·6	86·7	—	—	31·3	64°·9	77°·0	38°·0	—	85°·5	—	—	JR.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutu
76·6	78·6	81·8	—	77·2	72·3	31·2	64°·0	74°·4	41°·6	84°	82°·5	32°·0	9°·6	JR.
77·7	79·8	81·6	85·9	—	—	29·3	72°·8	70°·9	36°·3	75°	74°	32°·2	4°·1	JR.
81·1	83·4	82·2	85·2	79·7	73·6	38·6	—	—	—	—	81°	—	—	JR.
74·4	74·0	76·9	84·3	—	—	30·5	—	—	—	—	80°	—	—	JR.
77·7	82·6	—	—	—	—	27·0	65°·2	75°·3	39°·5	83°	81°·5	31°·8	7°·7	Coronal and sagittal sutu
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutu
82·8	86·2	88·1	—	—	—	—	—	—	—	—	87°·5	—	—	Coronal and sagittal sutu
79·6	—	—	—	—	—	—	66°·0	74°·0	40°·0	—	—	—	—	Coronal suture beginning
83·3	82·6	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal sutu
81·3	81·6	85·5	86·5	86·5	78·6	28·2	—	—	—	81°·5	80°	—	—	
87·5	82·9	—	—	78·4	72·6	38·5	—	—	—	82°	79°	—	—	Coronal suture beginning
87·3	87·5	92·3	95·7	—	—	—	—	—	—	—	79°	—	—	
88·1	82·0	—	—	—	—	—	66°·9	73°·4	39°·7	—	96°	—	—	Sagittal suture beginning
75·5	—	—	—	—	—	22·7	—	—	—	—	—	—	—	Sagittal suture beginning
83·1	80·7	83·8	86·6	98·1	90·7	31·7	65°·6	70°·1	44°·3	84°·5?	82°	29°·9?	14°·4?	JL.
87·7	93·2	—	—	73·8	67·7	33·8	72°·1	66°·2	41°·7	79°	77°·5	28°·9	12°·8	Coronal and sagittal sutu
91·8	92·7	—	—	86·2	80·6	35·1	64°·1	69°·5	46°·4	—	—	—	—	JL.
85·4	81·3	90·6	92·8	67·7	64·1	27·4	71°·8	68°·1	40°·1	71°	70°	37°·2	2°·9	
—	81·2	—	—	—	—	—	—	—	—	—	—	—	—	Coronal suture beginning
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture obliterat
83·1	77·1	88·5	—	74·0	67·0	41·9	68°·6	71°·1	40°·3	78°	76°	33°·4	6°·9	All sutures partly closed
86·8	86·1	90·5	93·9	—	—	—	67°·0	70°·5	42°·5	93°	91°·5	20°·0	22°·5	Sagittal suture beginnin
—	83·3	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture closed ar
92·5	89·8	95·1	99·5	—	—	—	69°·6	68°·9	41°·5	81°	78°·5	29°·4	12°·1	Coronal and sagittal sut
87·9	89·5	—	95·0	—	—	—	63°·6	74°·9	41°·5	—	83°	—	—	All sutures closing. 2 m
79·8	78·7	82·4	82·8	71·1	64·7	40·6	61°·8	77°·8	40°·4	—	86°	—	—	Coronal and sagittal sut
—	—	—	—	—	—	—	—	—	—	—	—	—	—	
74·3	78·3	—	—	78·5	72·1	27·9	61°·5	79°·7	38°·8	—	85°	—	—	Coronal suture partly ob
—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures obliterated.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture closed.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Teeth worn.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal and coronal su
73·9	79·0	78·1	82·7	93·1	87·1	31·0	63°·1	74°·7	42°·2	—	83°·5	—	—	JR.
—	—	—	—	—	—	—	—	—	—	—	—	—	—	All sutures beginning to
77·0	76·2	82·7	83·5	85·4	75·8	—	—	—	—	—	—	—	—	Coronal and sagittal sut
83·6	87·4	—	—	—	—	—	63°·0	71°·5	45°·5	80°?	78°	37°·0?	8°·5?	Sagittal suture beginni
80·0	81·9	81·0	87·2	88·0	82·7	41·4	66°·3	72°·2	41°·5	95°·5	94°	18°·2	23°·3	One premolar lost befor
73·5	71·5	74·2	76·8	—	—	—	61°·4	79°·5	39°·1	—	94°	—	—	All sutures closed and p
74·0	76·7	—	—	84·4	74·5	30·3	75°·3	66°·4	38°·3	88°·5?	85°·5	16°·2?	22°·1?	JR.
82·6	83·7	88·9	90·9	—	—	24·7	—	—	—	—	—	—	—	
69·5	71·3	72·5	73·3	—	—	25·0	—	—	—	—	—	—	—	Teeth worn. JR.
88·5	88·4	94·4	94·8	74·1	72·7	33·7	70°·5	71°·4	38°·1	78°	77°	31·5°	6°·6	Coronal suture closed, s
85·1	82·6	89·7	91·7	72·8	70·9	26·0	—	—	—	75°	73°	—	—	Sagittal suture beginni
79·9	84·2	82·3	85·2	80·2	74·0	43·3	66°·2	71°·8	42°·0	79°·5	78°	34°·3	7°·7	All sutures closed and p
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture beginni
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Sagittal suture obliterat
79·9	81·4	86·8	91·4	—	—	—	70°·5	68°·7	40°·8	78°·5	75°·5	31°·0	9°·8	One molar lost before c
—	—	—	—	—	—	—	—	—	—	—	—	—	—	Coronal and sagittal su
74·4	77·4	—	—	78·2	72·3	34·8	70°·5	68°·4	41°·1	76°	74°	33°·5	7°·6	Coronal suture beginni
75·0	75·6	—	—	—	—	—	72°·0	70°·8	37°·2	82°	79°	26°·0	11°·2	Coronal and sagittal su
76·2	82·3	81·8	—	87·1	76·6	30·0	69°·1	72°·0	38°·9	82°	78°	28°·9	10°·0	Coronal suture beginni
—	79·7	—	—	—	—	—	—	—	—	—	—	—	—	All sutures closed.
82·7	79·7	84·0	87·1	84·7	82·2	29·6	68°·4	70°·9	40°·7	—	—	—	—	Lambdoid suture begin
—	—	—	—	—	—	—	—	—	—	81°	79°	—	—	All sutures closed. Tee

Remarks
All sutures closed, coronal nearly obliterated.
JR.
Coronal and sagittal sutures beginning to close. Teeth worn. J=.
JR.
JR.
JR.
JR.
Coronal and sagittal sutures beginning to close. JR.
Coronal and sagittal sutures closed, lambdoid closing. JL.
Coronal and sagittal sutures beginning to close. 2 molars lost before death. JL.
Coronal suture beginning to close.
Coronal and sagittal sutures beginning to close. Teeth worn.
——
Coronal suture beginning to close. Teeth worn.
——
Sagittal suture beginning to close.
Sagittal suture beginning to close. JR.
JL.
Coronal and sagittal sutures partly closed, lambdoid closing. Teeth worn and 2 molars lost before death. JL.
JL.
——
Coronal suture beginning to close. Teeth worn and 1 molar lost before death. JR.
Sagittal suture obliterated, coronal and lambdoid partly obliterated. JR.
All sutures partly closed. JL.
Sagittal suture beginning to close. 5 or more teeth lost before death.
Sagittal suture closed and partly obliterated. J=.
Coronal and sagittal sutures partly obliterated, lambdoid closed. JR.
All sutures closing. 2 molars lost before death. JR.
Coronal and sagittal sutures obliterated, lambdoid closed. JR.
——
Coronal suture partly obliterated, sagittal and lambdoid closed. Teeth worn.
All sutures obliterated. JR.
Sagittal suture closed.
Teeth worn.
Sagittal and coronal sutures beginning to close. Teeth worn and 2 or more lost before death.
JR.
All sutures beginning to close.
Coronal and sagittal sutures beginning to close. JL.
Sagittal suture beginning to close. 1 or more teeth lost before death. JR.
One premolar lost before death.
All sutures closed and partly obliterated. 4 teeth lost before death. JR.
JR.
——
Teeth worn. JR.
Coronal suture closed, sagittal closing. Teeth worn. JL.
Sagittal suture beginning to close. J=.
All sutures closed and partly obliterated. JL.
Sagittal suture beginning to close.
Sagittal suture obliterated. Teeth worn and 2 lost before death. JL.
One molar lost before death. JR.
Coronal and sagittal sutures beginning to close. JL.
Coronal suture beginning to close. Teeth worn. JR.
Coronal and sagittal sutures beginning to close. Teeth worn and 3 molars lost before death.
Coronal suture beginning to close. Teeth worn. J=.
All sutures closed.
Lambdoid suture beginning to close. JL.
All sutures closed. Teeth worn. JL.

To face p. 284



Typical Male Kerma Skull (No. 27). *Norma facialis.*

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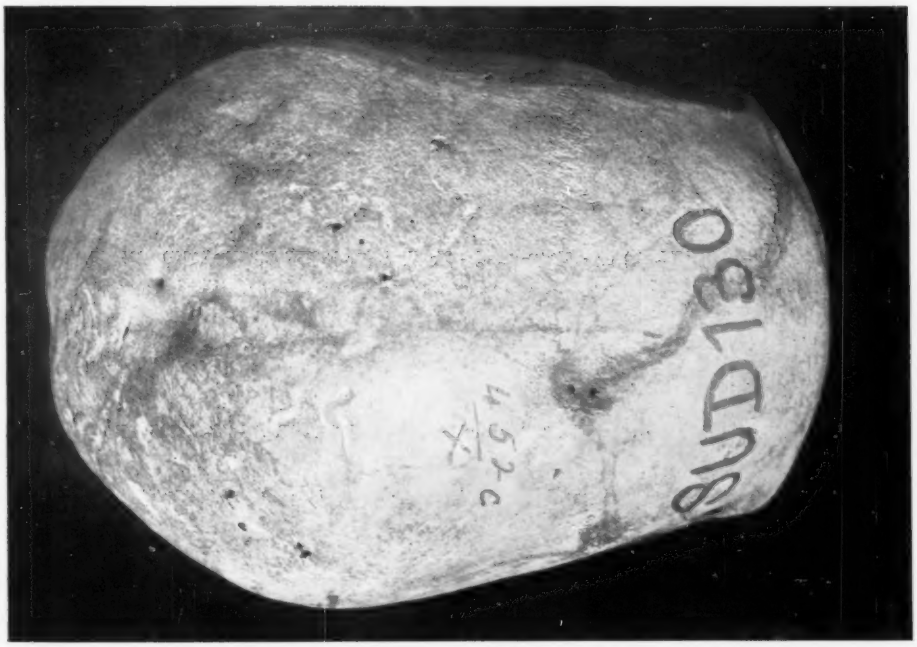
A Typical Male Kerma Skull (No. 27). *Norma lateralis.*

(*a*) Typical male skull.



(*b*) Male skull showing holes presumed to have been made by insects after death.

Male Kerma Skulls. ***Normae verticales.***

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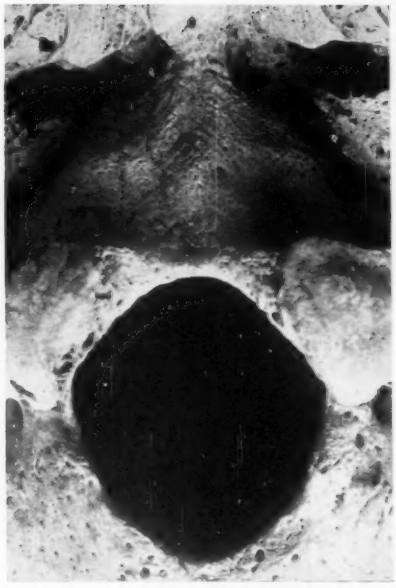
(*a*) Male skull (No. 74), with wounds on the parietal and malar bones.

(*b*) Female skull (No. 153), with large Wormian bones between the temporal squama and parietal bone.

Anomalous Kerma Skulls.

(*a*) Female skull (No. 274), showing diseased areas on either side of the lambda.



(*b*) Male skull (No. 36), showing constriction of the basi-occipital where it unites with the sphenoid.

(*c*) Male skull (No. 64), showing a wound below the orbit.

(*d*) Female skull (No. 216), showing failure of the tympanic elements to unite.

Anomalous Kerma Skulls.

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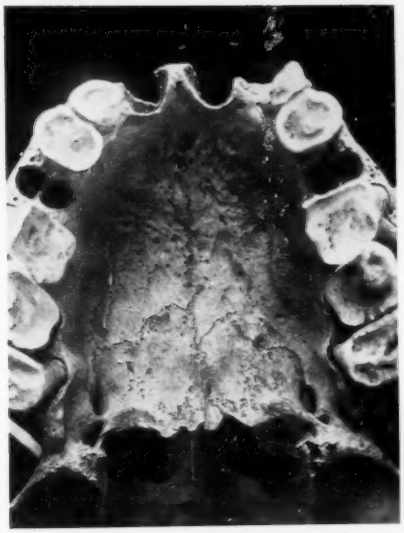
B

(a
o

(c)
f

(*a*) Male skull (No. 92), with the sockets of two teeth other than molars lacking.



(*b*) Female skull (No. 216), with no sockets for lateral incisors.

(*c*) Female skull (No. 155), with no sockets for the left canine and right third molar.

(*d*) Male skull (No. 107), with sockets for three incisors only.

Anomalous Palates of Kerma Skulls.

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ON THE LIKELIHOOD THAT ONE UNKNOWN PROBABILITY EXCEEDS ANOTHER IN VIEW OF THE EVIDENCE OF TWO SAMPLES.

By WILLIAM R. THOMPSON. From the Department of Pathology, Yale University.

Section 1.

In elaborating the relations of the present communication interest was not centred upon the interpretation of particular data, but grew out of a general interest in problems of research planning. From this point of view there can be no objection to the use of data, however meagre, as a guide to action required before more can be collected; although serious objection can otherwise be raised to argument based upon a small number of observations. Indeed, the fact that such objection can never be eliminated entirely—no matter how great the number of observations—suggested the possible value of seeking other modes of operation than that of taking a large number of observations before analysis or any attempt to direct our course. This problem is more general than that treated in *Section* 2, and is directly concerned with any case where probability criteria may be established by means of which we judge whether one mode of operation is *better* than another in some given sense or not.

Thus, if, in this sense, P is the probability estimate that one *treatment* of a certain class of individuals is *better* than a second, as judged by data at present available, then we might take some monotone increasing function of P, say $f_{(P)}$, to fix the fraction of such individuals to be treated in the *first manner*, until more evidence may be utilised, where $0 \leqslant f_{(P)} \leqslant 1$; the remaining fraction of such individuals $(1 - f_{(P)})$ to be treated in the *second manner*; or we may establish a probability of treatment by the two methods of $f_{(P)}$ and $1 - f_{(P)}$, respectively. If such a discipline were adopted, even though it were not the best possible, it seems apparent that a considerable saving of individuals otherwise sacrificed to the inferior treatment might be effected. This would be important in cases where either the rate of accumulation of data is slow or the individuals treated are valuable, or both.

If we arbitrarily decide to eliminate the second treatment in favour of the first at this time, then the expectation of sacrifice to the inferior treatment would be $(1 - P)$ for *all* subsequently treated individuals; whereas, if, for example, we take $f_{(P)} = P$, the expectation of such sacrifice would be *temporarily*

$$P(1 - P) + (1 - P)P = 2PQ,$$

where $Q = 1 - P$. Obviously, $2PQ \leqslant \frac{1}{2}$ and, if $P \neq \frac{1}{2}$, then $2PQ < \frac{1}{2}$; whence a saving is made in contrast to the so-called *alternate case method.* In the *long run*, if a real preference exists between the two *treatments*, the expected saving by continued application of this method of apportionment rather than by making immediate final decision is sensibly $1 - P$ of individuals subsequently treated.

Obviously, if we are to operate in this manner, we need methods of evaluation of P for *small* as well as *large* numbers of observations. In the latter case many approximate methods are available in all fields although bounds to approximation have not been considered generally.

In *Section* 2 a sampling problem is treated, which is equivalent to a special case, where we are to judge between two rival treatments upon the basis of the probability of occurrence of a given critical event following such treatment. These probabilities are assumed unknown, but denoted by $\tilde{p}_1$ and $\tilde{p}_2$; and it is assumed that, independently for each of these, *a priori* $\tilde{p}_i$ is equally likely to lie in either of any two equal intervals in its possible range, (0, 1). Our available experience consists solely of the data:

Of n_1 individuals treated by the first method, r_1 experienced the critical event and s_1 did not; and of n_2 treated by the second, r_2 and s_2 were the corresponding numbers with respect to the critical event.

In any given case it must be decided whether these requirements are met or not, and whether we may apply the well-known Principle of Bayes to convert the problem to the form of *Section* 2. Statistical criteria are often employed, however, in situations in which certain deviations from the conditions required in their development can be tolerated, when a better procedure is not available.

Section 2.

Consider the case of two infinite populations for which the unknown probabilities of occurrence of a given critical event are $\tilde{p}_1$ and $\tilde{p}_2$, and the probability of obtaining a sample containing r critical occurrences and s failures in $n = r + s$ trials is $\binom{n}{r} . \tilde{p}_i^r (1 - \tilde{p}_i)^s$, where $i = 1, 2$, respectively. Furthermore, assume that one sample has been drawn at random from each population, the respective values of r and s being denoted by r_i and s_i (where $i = 1, 2$) and $n_i = r_i + s_i$; and that *independently* for $i = 1$ or 2 the probability that $\tilde{p}_i$ lies in the interval $(p, p + dp)$ is $P^{(i)}_{p, p+dp}$, where

$$(1) \qquad P^{(i)}_{p, p+dp} = \frac{\int_p^{p+dp} \binom{n}{r} . p^r . q^s . dp}{\int_0^1 \binom{n}{r} . p^r . q^s . dp},$$

where $q = 1 - p$, $r = r_i$, $s = s_i$, and $n = r + s$. Then

$$(2) \qquad P^{(i)}_{p, 1} = \frac{(n+1)!}{r! . s!} \int_p^1 p^r . q^s . dp = \sum_{a=0}^{r} \binom{n+1}{a} . p^a . q^{n+1-a};$$

the last expression having been indicated by K. Pearson* in this relation. In the notation employed by him and by Müller† we may write

$$1 - P^{(i)}_{p,1} = P^{(i)}_{0,p} = I_p(u, v) = \frac{B_p(u, v)}{B_1(u, v)} = 1 - I_q(v, u), \tag{3}$$

where $u = r_i + 1$ and $v = s_i + 1$. The object of the present communication is to give a reduced‡ rational algebraic evaluation of the probability $(P_{\tilde{p}_2 > \tilde{p}_1})$ that for the postulated systems $\tilde{p}_2$ exceed $\tilde{p}_1$, and to indicate certain relations between its value (later designated by $\Psi_{(r_1, s_1, r_2, s_2)}$) and the sum of the first $r_2 + 1$ terms of a *hypergeometric series* which has appeared in the work of K. Pearson§|| as well as in the *Incomplete B-* and *I-functions**† of (3).

Obviously, we may write

$$P_{\tilde{p}_2 > \tilde{p}_1} = \frac{(n_1+1)!}{r_1!\, s_1!} \int_0^1 p_1^{r_1} \cdot q_1^{s_1} \cdot \frac{(n_2+1)!}{r_2!\, s_2!} \int_{p_1}^1 p_2^{r_2} \cdot q_2^{s_2} \cdot dp_2 \cdot dp_1 \tag{4}$$ (i)

$$= \frac{(n_1+1)!}{r_1!\, s_1!} \int_0^1 \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} \cdot p^{r_1+\alpha} \cdot q^{s_1+n_2+1-\alpha} \cdot dp$$ (ii)

$$= \frac{(n_1+1)!}{r_1!\, s_1!} \cdot \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} \cdot \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{(n_1+n_2+2)!}$$ (iii)

$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!} \cdot \sum_{\alpha=0}^{r_2} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{r_1!\,\alpha!\, s_1!\,(n_2+1-\alpha)!}$$ (iv)

$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!} \cdot \sum_{\alpha=0}^{r_2} \frac{(r_1+r_2-\alpha)!\,(s_1+s_2+1+\alpha)!}{r_1!\,(r_2-\alpha)!\, s_1!\,(s_2+1+\alpha)!},$$ (v)

whence we have

$$P_{\tilde{p}_2 > \tilde{p}_1} = \frac{\sum_{\alpha=0}^{r_2} \binom{r_1+r_2-\alpha}{r_1} \cdot \binom{s_1+s_2+1+\alpha}{s_1}}{\binom{n_1+n_2+2}{n_1+1}}, \tag{5}$$

where, of course, $n_i = r_i + s_i$.

Now, it is obvious that $P_{\tilde{p}_2 > \tilde{p}_1} = P_{\tilde{p}_1 < \tilde{p}_2} = P_{\tilde{q}_1 > \tilde{q}_2} = P_{\tilde{q}_2 < \tilde{q}_1}$, where $\tilde{q}_i = 1 - \tilde{p}_i$, and thus in similar manner we have

$$P_{\tilde{p}_2 > \tilde{p}_1} = \frac{\sum_{\alpha=0}^{s_1} \binom{s_1+s_2-\alpha}{s_2} \cdot \binom{r_1+r_2+1+\alpha}{r_2}}{\binom{n_1+n_2+2}{n_1+1}}. \tag{6}$$

Furthermore, $P_{\tilde{p}_2 > \tilde{p}_1} = 1 - P_{\tilde{p}_1 > \tilde{p}_2}$, as the probability that $\tilde{p}_1$ is exactly equal to $\tilde{p}_2$ is zero by hypothesis. Hence we have two other similar sums which may be used with this difference relation to evaluate the probability under consideration.

* Pearson, Karl: *Biometrika*, Vol. XVI (1924), pp. 202—203.

† Müller, J. H.: *Biometrika*, Vol. XXII (1930—31), pp. 284—297.

‡ The earliest work directed to this end is discussed by Todhunter. Cf. *A History of the Mathematical Theory of Probabilty*. Cambridge and London (1865), pp. 419—420.

§ Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

|| Pearson, Karl: *Biometrika*, Vol. XX^A (1928), pp. 149—174.

In actual evaluation we may make use of the well-known *pyramid form* of tabulation of the binomial coefficients,

(7)

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1
	5		10		10		5	
6		15		20		15		6
	•		•		•		•	
•		•		•		•		•

which is readily constructed by the property that each entry except those on the boundaries (which are always unity) is the sum of the two nearest entries of the row next above*. The corresponding factors of the successive terms in the sum to be evaluated may be found in order on given diagonals, ascending in the case of the first and descending in the case of the second factor. The bounding diagonals of units may be deleted in practice (as may all entries to the right of the middle column). Then the first factor of the first term of the sum in (5) is the r_1-th entry in the row whose first entry (at the left) is r_1+r_2, and the corresponding factors of successive terms lie successively above on the diagonal through the first and parallel to the left-hand boundary. Similarly, the other factors are found, but proceeding in the opposite direction on the appropriate diagonal.

Now, for any positive number, x, let $Q_{(x)} = x^{x+\frac{1}{2}}$. Then, by Stirling's formula, we have

$$m\,! = Q_{(m)} \cdot \sqrt{2\pi} \cdot e^{-m} \cdot e^{\frac{\theta}{12m}}, \tag{8}$$

where $0 \leqslant \theta \leqslant 1$; whence (4)(v) gives

$$P_{\tilde{p}_2 > \tilde{p}_1} = \frac{Q_{(n_1+1)} \cdot e^{w}}{Q_{(n_1+n_2+2)} \cdot Q_{(r_1)} \cdot Q_{(s_1)}} \cdot \sum_{\alpha=0}^{r_2} \binom{n_2+1}{r_2-\alpha} Q_{(r_1+r_2-\alpha)} \cdot Q_{(s_1+s_2+1+\alpha)}, \tag{9}$$

where

$$|12w| < \frac{1}{r_1} + \frac{1}{s_1} + \frac{1}{n_1+1};$$

which may be used to advantage in approximation of $P_{\tilde{p}_2 > \tilde{p}_1}$ when r_1 and s_1 are *large* and r_2 *small*.

Now we may define $\psi_{(r_1, s_1, r_2, s_2)}$ for any four rational integers (not negative) as identical with the right member of (5), where $n_i = r_i + s_i > 0$. Then we have shown several equivalent expressions of the same function and that

$$\psi_{(r_1, s_1, r_2, s_2)} = \psi_{(s_2, r_2, s_1, r_1)} = 1 - \psi_{(s_1, r_1, s_2, r_2)}. \tag{10}$$

* Glaisher, J. W. L.: "A Table of Binomial-Theorem Coefficients," *Messenger of Mathematics*, Vol. 47 (1917), pp. 97—107.

From the conditions stated it may be expected that if we set $p = \frac{r_1}{n_1}$ and $q = 1 - p$, then (provided $0 < p < 1$),

$$\lim_{n_1 \to \infty} \frac{\psi_{(r_1, s_1, r_2, s_2)}}{I_q(s_2 + 1, r_2 + 1)} = 1. \tag{11}$$

That this is true may be verified if we exclude the cases, $p = 0, 1$. Further bounds to approximation of this limit by the ratio ($\bar{R}$) of these functions for given values of n_1 may be found as follows:

By (4)(iv) we may write

$$\psi_{(r_1, s_1, r_2, s_2)} = \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} \cdot \frac{\frac{(r_1+\alpha)!}{r_1!} \cdot \frac{(s_1+n_2+1-\alpha)!}{s_1!}}{\frac{(n_1+n_2+2)!}{(n_1+1)!}}, \tag{12}$$

and by (2) and (3), introducing the appropriate values of $r = r_2$ and $s = s_2$, we have

$$I_q(s_2 + 1, r_2 + 1) = \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} \cdot p^{\alpha} \cdot q^{n_2+1-\alpha}, \tag{13}$$

where $p = \frac{r_1}{n_1}$ and $q = \frac{s_1}{n_1}$. Obviously, therefore, as all terms of both sums in (12) and (13) are positive, if we exclude the special cases where p or $q = 0$, $\bar{R}$ is bounded by the greatest and least values attainable for the ratio of a term in the sum of (12) to the corresponding term in (13). Thus we may define

$$\omega_1 = \text{Min.} \left[\left(\frac{r_1+1}{r_1} \right)^{\alpha} \left(\frac{s_1+1}{s_1} \right)^{n_2+1-\alpha} \left(\frac{n_1}{n_1+n_2+2} \right)^{n_2+1} \right] \tag{14}$$

and

$$\omega_2 = \text{Max.} \left[\left(\frac{r_1+\alpha}{r_1} \right)^{\alpha} \left(\frac{s_1+n_2+1-\alpha}{s_1} \right)^{n_2+1-\alpha} \left(\frac{n_1}{n_1+2} \right)^{n_2+1} \right]$$

for $0 \leqslant \alpha \leqslant r_2$; and, obviously, then

$$\omega_1 < \bar{R} < \omega_2, \text{ and } \lim_{n_1 \to \infty} [\bar{R}] = 1. \tag{15}$$

In the excluded cases it is also readily verified that

$$\lim_{n_1 \to \infty} [\psi_{(r_1, s_1, r_2, s_2)}] = I_q(s_2 + 1, r_2 + 1). \tag{16}$$

The relation of this function to the sum of a given number of consecutive terms of a *hypergeometric series* is particularly interesting in view of the occurrence of such series in the investigations of K. Pearson*†. By (4)(iv) we may write:

$$\psi_{(r_1, s_1, r_2, s_2)} = \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!} \cdot \sum_{\alpha=0}^{r_2} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}, \tag{17}$$

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

† Pearson, Karl: *Biometrika*, Vol. xx^{A} (1928), pp. 149—174.

which is the sum of the first r_2+1 terms of a *hypergeometric series* multiplied by a constant. Similarly, we may write

$$(18)\quad \psi_{(s_1,r_1,s_2,r_2)} = \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!}\cdot\sum_{\alpha=0}^{s_2}\frac{(s_1+\alpha)!\,(r_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}$$

$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!}\cdot\sum_{\alpha=0}^{s_2}\frac{(r_1+r_2+1+\alpha)!\,(s_1+s_2-\alpha)!}{(r_2+1+\alpha)!\,(s_2-\alpha)!}$$

$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!}\cdot\sum_{\alpha=r_2+1}^{n_2+1}\left[\frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}\right],$$

obviously (as $n_2 = r_2+s_2$); and by previous demonstration

$$(19)\qquad \psi_{(r_1,s_1,r_2,s_2)} + \psi_{(s_1,r_1,s_2,r_2)} = 1,$$

whence

$$(20)\qquad \frac{(n_1+n_2+2)!\,r_1!\,s_1!}{(n_1+1)!\,(n_2+1)!} = \sum_{\alpha=0}^{n_2+1}\frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}.$$

The last relation, demonstrated above by independent proof, has been established previously by Pearson*† (with different notation). Thus we may regard $\psi_{(r_1,s_1,r_2,s_2)}$ as defined by the identity

$$(21)\qquad \psi_{(r_1,s_1,r_2,s_2)} \equiv \frac{\sum_{\alpha=0}^{r_2}\frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}}{\sum_{\alpha=0}^{n_2+1}\frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}},$$

extending the domain of definition to include the value, $r_2 = n_2+1$; but retaining the restrictions, $n_i = r_i+s_i \geqslant 0$, and that -1 be the least value of r_1, s_1, r_2, and s_2 (only one of which shall be admitted to be negative). Then, by this extension, we have

$$(22)\qquad \psi_{(r_1,s_1,0,0)} \equiv \frac{s_1+1}{r_1+s_1+2},\ \text{and}\ \psi_{(r_1,s_1,n_2+1,-1)} \equiv 1,$$

which lie outside the domain of the initial discussion, and we extend to the new domain the relation of (10) *formally*; i.e.

$$(23)\qquad \psi_{(r,s,r',s')} \equiv \psi_{(s',r',s,r)} \equiv 1 - \psi_{(s,r,s',r')}.$$

K. Pearson† has considered the problem of likelihood of various values of R and S, the number of *marked* members and *unmarked* members, respectively, in a finite universe of aggregate number, $N = R+S$; assuming N fixed and all values of R, $S \geqslant 0$ equally likely *a priori* and that our sole experience from which judgment is to be made is that a *random sample* has been drawn containing exactly r marked and s unmarked members (R and S being used here in place of Pearson's p and q to avoid confusion). Then by (iii) and (iv) of the article† just mentioned, we have a means of evaluating the probability, $\bar{P}_R$, that the universe contains no more than R marked members by the relation,

$$(24)\qquad \bar{P}_R \equiv \psi_{(r,s,R-r,N-R-s-1)},$$

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

† Pearson, Karl: *Biometrika*, Vol. xx^A (1928), pp. 149—174.

which may be verified readily. Similarly, in the case of the problem considered earlier by Pearson*—having drawn one random sample from a certain infinite population, the sample containing exactly r' *marked* and s' *unmarked* members, we are required to find the probability (under the given conditions) that if we draw another random sample of n'' individuals from the same population it will contain no more than r'' marked members—the required value is given for $r'' \leqslant n''$ by

$$\psi_{(r', s', r'', n''-r''-1)}.$$

In the tabulation of values of $\psi_{(r, s, r', s')}$ for ascending values of the arguments the work may be greatly simplified by certain relations in addition to those given in (22) and (23) in much the same manner as the binomial coefficients may be tabulated† by mere summation of two values already given. To this end let us examine two functions defined by

$$N_{(r, s, r', s')} \equiv \sum_{\alpha=0}^{r'} \binom{r + r' - \alpha}{r} \binom{s + s' + 1 + \alpha}{s} \tag{25}$$

and $$D_{(r, s, r', s')} \equiv D_{(n, n')} \equiv \binom{n + n' + 2}{n + 1} \quad \text{for } n = r + s, \text{ and } n' = r' + s'.$$

Obviously, by the original definition of $\psi_{(r, s, r', s')}$, extended in (21), (22) and (23), then

$$\psi_{(r, s, r', s')} \equiv \frac{N_{(r, s, r', s')}}{D_{(r, s, r', s')}}, \tag{26}$$

where we extend the definition of (25) for r, s, r', $s' \geqslant 0$ by

$$N_{(r, s, -1, s')} \equiv 0, \text{ and } N_{(r, s, r', -1)} \equiv D_{(r+s, r'-1)}, \tag{27}$$

and by (26) and (23) we have

$$N_{(r, s, r', s')} \equiv N_{(s', r', s, r)} \equiv D_{(r+s, r'+s')} - N_{(s, r, s', r')}, \tag{28}$$

as it is obvious that $D_{(n, n')} \equiv D_{(n', n)}$. Furthermore, by the well-known relation,

$$\binom{a}{b} \equiv \binom{a - 1}{b - 1} + \binom{a - 1}{b}, \tag{29}$$

$$D_{(n, n')} \equiv D_{(n, n'-1)} + D_{(n-1, n')},$$

we have for $s > 0$ in (25)

$$N_{(r, s, r', s')} \equiv \sum_{\alpha=0}^{r'} \binom{r + r' - \alpha}{r} \left[\binom{s + s' + \alpha}{s - 1} + \binom{s + s' + \alpha}{s} \right], \tag{30}$$

whence we have in any case under its definition, obviously, by (20), (27) and (28), the identities

$$\begin{aligned} N_{(r, s, r', s')} &\equiv N_{(r, s-1, r', s')} + N_{(r, s, r', s'-1)} \\ &\equiv N_{(r-1, s, r', s')} + N_{(r, s, r'-1, s')}. \end{aligned} \tag{31}$$

By (25) and (29), obviously, the same relation holds for the D-function; and we may write

$$\psi_{(r, s, r', s')} \equiv \frac{N_{(r, s-1, r', s')} + N_{(r, s, r', s'-1)}}{D_{(r, s-1, r', s')} + D_{(r, s, r', s'-1)}}. \tag{32}$$

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365–378.

† Glaisher, J. W. L.: "A Table of Binomial-Theorem Coefficients," *Messenger of Mathematics*, Vol. 47 (1917), pp. 97–107.

By means of these relations it is evident that it suffices to tabulate the N and the D-functions by additions of corresponding pairs of values already listed or readily obtained by the relations of (28) if we proceed from the lowest values of the arguments upward; and we need list values only for the cases where $r+s \geqslant r'+s'$ and $r \geqslant s$ where all four variables may be restricted to positive values, as by (25) we have

$$N_{(r,s,0,s')} \equiv D_{(s-1,s')}. \tag{33}$$

By (25) we may write

$$N_{(r,s,r',s')} \equiv \sum_{a=0}^{r'} D_{(r-1,r'-1-a)} \cdot D_{(s-1,s'+a)}, \tag{34}$$

which is of value if $D_{(n,n')}$ is tabulated for $n \geqslant n'$ through ascending values of n and n'. This may be done rapidly by means of (29) and the relation $D_{(n,n')} \equiv D_{(n',n)}$ and is more convenient than the pyramid form of the corresponding binomial coefficients. In the tabulation we may restrict n and n' to the positive integers, employing the relation $D_{(n,0)} \equiv n+2$. A short table of the N and D functions is appended as an illustration, the required probability being the ratio of these corresponding values.

Section 3.

If a system of operation such as suggested in *Section* 1 were adopted extensively for the case considered in *Section* 2, reference to values of $\psi_{(r,s,r',s')}$ for small values of the arguments should be required frequently; and, accordingly, a simple method of formation of a table of these would be valuable. The method given in *Section* 2 seems to serve this purpose; and, in conjunction with the relations of (28), many values need not be listed. The short table is given merely in illustration. It really deals completely with all cases of $n, n' \leqslant 5$ although certain cases are not listed where the values are readily obtainable from those given and (28). The several general evaluations at the head of the table would permit deletion of many more, e.g. any instance where one of the four arguments is zero; but they have been retained for illustrative purposes.

The function, $D_{(n,n')} \equiv \binom{n+n'+2}{n+1}$, is readily tabulated in a convenient form for increasing values of $n \geqslant n' \geqslant 1$, as has been mentioned under (34) above by adding to $n+2$ successively the values of $D_{(n-1,n')}$, already listed, and taking a sub-total after each addition, and finally doubling the last sub-total. These sub-totals and the final double are the required successive values of $D_{(n,n')}$. The value of such a table extends far beyond that of the immediate problem; and, by means of it and relation (34), $\psi_{(r,s,r',s')}$ may be calculated rapidly or approximated with any required precision for considerably higher values of the arguments than it may be convenient to have tabulated $N_{(r,s,r',s')}$. In accord with some prescribed tolerance and limited extent, a table of approximate values of the D-function could be made with greater ease, but apparently not readily extensible within the same relative tolerance without revision. Just how far these tables should extend would depend upon demands for their use. All questions as to approximate methods should be decided

Short Table of $N_{(r,s,r',s')}$ and $D_{(n,n')}$ ($n = r + s$, and $n' = r' + s'$)

r	s	r'	s'	N	D
a	b	c	d	$\overline{N}$	$\overline{D}$
b	a	d	c	$\overline{D}-\overline{N}$	$\overline{D}$
d	c	b	a	$\overline{N}$	$\overline{D}$
r	s	-1	s'	0	
r	0	0	0	1	$r+2$
r	0	0	s'	1	
r	0	1	s'	$r+2$	
r	s	0	0	$s+1$	$n+2$
r	s	0	s'	$D_{(n-1,s')}$	
0	0	-1	0	0	1
0	0	0	0	1	2
1	0	0	0	1	3
0	1	0	0	2	
1	0	0	1	1	6
1	0	1	0	3	
2	0	0	1	1	10
2	0	1	0	4	
1	1	0	1	3	
2	0	0	2	1	20
2	0	1	1	4	
3	0	0	1	1	15
3	0	1	0	5	
2	1	0	1	3	
2	1	1	0	9	
3	0	0	2	1	35
3	0	1	1	5	
3	0	2	0	15	
2	1	0	2	4	
2	1	1	1	13	
2	1	2	0	25	
3	0	0	3	1	70
3	0	1	2	5	
3	0	2	1	15	
2	1	1	2	17	
4	0	0	1	1	21
4	0	1	0	6	
3	1	0	1	3	
3	1	1	0	11	
2	2	0	1	6	
4	0	0	2	1	56
4	0	1	1	6	
4	0	2	0	21	
3	1	0	2	4	
3	1	1	1	16	
3	1	2	0	36	
2	2	0	2	10	
2	2	1	1	28	
4	0	0	3	1	126
4	0	1	2	6	
4	0	2	1	21	
4	0	3	0	56	
3	1	0	3	5	
3	1	1	2	21	
3	1	2	1	51	
3	1	3	0	91	
2	2	0	3	15	
2	2	1	2	45	

r	s	r'	s'	N	D
4	0	0	4	1	252
4	0	1	3	6	
4	0	2	2	21	
4	0	3	1	56	
3	1	1	3	26	
3	1	2	2	66	
5	0	0	1	1	28
5	0	1	0	7	
4	1	0	1	3	
4	1	1	0	13	
3	2	0	1	6	
3	2	1	0	18	
5	0	0	2	1	84
5	0	1	1	7	
5	0	2	0	28	
4	1	0	2	4	
4	1	1	1	19	
4	1	2	0	49	
3	2	0	2	10	
3	2	1	1	34	
3	2	2	0	64	
5	0	0	3	1	210
5	0	1	2	7	
5	0	2	1	28	
5	0	3	0	84	
4	1	0	3	5	
4	1	1	2	25	
4	1	2	1	70	
4	1	3	0	140	
3	2	0	3	15	
3	2	1	2	55	
3	2	2	1	115	
3	2	3	0	175	
5	0	0	4	1	462
5	0	1	3	7	
5	0	2	2	28	
5	0	3	1	84	
5	0	4	0	210	
4	1	0	4	6	
4	1	1	3	31	
4	1	2	2	91	
4	1	3	1	196	
4	1	4	0	336	
3	2	0	4	21	
3	2	1	3	81	
3	2	2	2	181	
3	2	3	1	301	
3	2	4	0	406	
5	0	0	5	1	924
5	0	1	4	7	
5	0	2	3	28	
5	0	3	2	84	
5	0	4	1	210	
4	1	1	4	37	
4	1	2	3	112	
4	1	3	2	252	
3	2	2	3	262	

by several statisticians in consultation at a time when a definite programme for the use of these methods is formed.

In (24) and the paragraph in which it stands is given the relation between the hypergeometrical series studied by Pearson*† and $\psi_{(r,s,r',s')}$ by means of which it is obvious that any approximation methods valid for estimation of $\overline{P}_R$ in (24) are equally valid for the estimation of the corresponding ψ-function, several of which have been suggested by K. Pearson. The I-function of Pearson is related to the ψ-function by (11) to (15) also; and another approximation of $\psi_{(r_1,s_1,r_2,s_2)}$ is given in (9) with indicated domain of validity.

A further treatment of the ψ-function and the method of apportionment will be provided in a later paper.

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.
† Pearson, Karl: *Biometrika*, Vol. xx^A (1928), pp. 149—174.

ON ASYMPTOTIC FORMULAE FOR THE HYPERGEOMETRIC SERIES.

I. HYPERGEOMETRIC SERIES IN WHICH THE FOURTH ELEMENT, x, IS UNITY.

By O. L. DAVIES, M.Sc.

The hypergeometric series $F(\alpha, \beta, \gamma, 1)$ arises frequently in problems of chance when samples are taken from a finite population. For instance, in a population of size M in which p individuals possess a certain character A, and $q\,(=M-p)$, not A, the chance of drawing a sample of size n in which there are r of A and $s\,(=n-r)$, not A, is clearly

$$\frac{n!}{r!\,s!}\,\frac{p}{M}\cdot\frac{p-1}{M-1}\cdots\frac{p-r+1}{M-r+1}\cdot\frac{q}{M-r}\cdot\frac{q-1}{M-r-1}\cdots\frac{q-s+1}{M-n+1} \quad \text{.....(1).}$$

Hence the distribution of r and s in repeated samples of size n is given by the terms of the series

$$N\cdot\frac{(M-n)!\,p!}{M!\,(p-n)!}\left[1+\frac{nq}{1!\,(p-n+1)}+\frac{n(n-1)\,q(q-1)}{2!\,(p-n+1)(p-n+2)}+\ldots\right] \quad \text{...(2),}$$

where N is the number of samples. If we write $n=-\alpha$, $q=-\beta$, $p+1-n=\gamma$, this series can be transformed into the usual hypergeometric form

$$N\frac{(\gamma-\beta-1)!\,(\gamma-\alpha-1)!}{(\gamma-\alpha-\beta-1)!\,(\gamma-1)!}\left[1+\frac{\alpha\beta}{1!\,\gamma}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}+\ldots\right]$$

$$=N\frac{(\gamma-\beta-1)!\,(\gamma-\alpha-1)!}{(\gamma-\alpha-\beta-1)!\,(\gamma-1)!}F(\alpha, \beta, \gamma, 1) \quad \text{.....(3).}$$

The chance of drawing in a sample of size n at least r individuals possessing character A is equal to the sum of the first $(r+1)$ terms of (2). This is what is meant by the *probability integral* of the series; its determination is of paramount importance.

It is well known that, when M is infinite, and if p and q now measure the chance of an individual being A or not A respectively, the distribution of r and s in repeated samples of size n is given by the terms of the expansion of the binomial $N(p+q)^n$. Its probability integral has been expressed by an incomplete Beta function*. When neither p nor q is very small and n fairly large, the distribution may be fitted very closely by a normal curve. If, however, either p or q is very small, the distribution may be represented by a Poisson series. This may also be

* Karl Pearson, *Biometrika*, Vol. xvi. pp. 202—3.

fitted quite closely by a normal curve if n be sufficiently large. The probability integrals of both these distributions, normal and Poisson, have been tabulated*.

Hypergeometric series of type $F(\alpha, \beta, \gamma, 1)$ may arise in at least two other ways†, namely:

(i) The proportional frequencies of drawing in successive samples of a fixed size n, r marked and $s\,(=n-r)$ unmarked individuals $(r=0, 1, 2, 3, \ldots, n)$, from a population having previously drawn a sample of size N with p marked and $q\,(=N-p)$ unmarked individuals, are given by the successive terms of the hypergeometric series

$$F(\alpha, \beta, \gamma, 1), \text{ where } \begin{cases} \alpha = -n, & \beta = q+1, \\ \gamma = -(p+n), & \epsilon = \gamma - \alpha - \beta - 1 = -(N+2). \end{cases}$$

(ii) A sample of size n with r marked and $s\,(=n-r)$ unmarked individuals is taken from a finite population of size N. The likelihoods of N having p marked $(p=0, 1, 2, \ldots, N)$ and $q\,(=N-p)$ unmarked individuals are given by the successive terms of the hypergeometric series

$$cF(\alpha, \beta, \gamma, 1),$$

where

$$\alpha = r+1, \quad \beta = -(N-n), \quad \gamma = -(N-r),$$

$$\epsilon = \gamma - \alpha - \beta - 1 = -(n+2),$$

$$F(\alpha, \beta, \gamma, 1) = c^{-1} = \frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)}.$$

An asymptotic expression for the remainder of the series $F(\alpha, \beta, \gamma, 1)$ after the $(s+1)$st term has been provided by M. J. M. Hill‡. It may be written in the form

$$R(\alpha, \beta, \gamma, s) = \frac{\Pi(s+1, \gamma-1)\,(s+1)^{\alpha+\beta-\gamma}\, f(\alpha, \beta, \gamma, s)}{\Pi(s+1, \alpha-1)\,\Pi(s+1, \beta-1)\,(\gamma-\alpha-\beta)},$$

where

$$f(\alpha, \beta, \gamma, s) = 1 + \frac{(\gamma-\alpha)(\gamma-\beta)}{(\gamma-\alpha-\beta+1)(\gamma+s+1)} + \frac{(\gamma-\alpha)(\gamma-\alpha+1)(\gamma-\beta)(\gamma-\beta+1)}{(\gamma-\alpha-\beta+1)(\gamma-\alpha-\beta+2)(\gamma+s+1)(\gamma+s+2)} + \ldots.$$

I have applied this formula to a number of particular hypergeometric series and found that in most cases the series $f(\alpha, \beta, \gamma, s)$ did not converge with sufficient rapidity to be of practical use for finding the sum of a number of *significant* terms of the hypergeometric series. However, the formula was very useful for evaluating the extreme tail of the series.

Professor Burton H. Camp§ has introduced a method for the approximate determination of the tail of a frequency distribution, continuous or discrete. His

* *Tables for Statisticians and Biometricians*, Part I, Tables II and LI, LII.

† Karl Pearson, *Philosophical Magazine*, March, 1907, pp. 365—378; *Biometrika*, Vol. V. 1907, pp. 172—175, Vol. XIII. 1920, pp. 1—16, Vol. XX^{A}. 1928, pp. 149—174.

‡ *Proc. Lond. Math. Soc.* Ser. 2, Vol. 5.

§ *Biometrika*, Vol. XVI. p. 163.

formula has been applied* to the series $F(\alpha, \beta, \gamma, 1)$, but it fails to give good results when the "stump" lies within $\pm 2\sigma$ of the mode, σ being the standard deviation of the series whose discrete terms are treated as a frequency distribution.

In this paper we shall obtain approximations to the sum of a finite number of terms of the series $F(\alpha, \beta, \gamma, 1)$ by fitting to it a Pearson-type curve. The question will be investigated of how closely the probability integral of the series may be represented by the probability integral of a Pearson-type curve.

The series

$$F(\alpha, \beta, \gamma, 1) = 1 + \frac{\alpha\beta}{1!\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} + \ldots,$$

when infinite, is convergent as long as $\gamma > \alpha + \beta$. The sth moment of the terms of the series, represented by a histogram with grouping unit "c," about a point c before the midordinate of the first block is

$$\left(1 \cdot 1^s + \frac{\alpha\beta}{1!\gamma} 2^s + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} 3^s + \ldots\right) c^s.$$

Applying Raabe's test, the condition of convergence of this more general series is found to be $\gamma > \alpha + \beta + s$, or $\epsilon > s - 1$, where $\epsilon = \gamma - \alpha - \beta - 1$. We cannot, therefore, have an infinite hypergeometric series with all its moments finite.

Convenient expressions for the moment coefficients of the above series, finite or infinite, have been found by Professor Karl Pearson. They are

$$N = \text{sum of series} = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad \epsilon > 0,$$

$$= \frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)} \qquad \epsilon < 0,$$

$$\mu_1' = c\frac{\alpha\beta}{\epsilon},$$

$$\mu_2 = \sigma^2 = \frac{c^2\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}{\epsilon^2(\epsilon-1)},$$

$$\mu_3 = \frac{c^3\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)(2\alpha+\epsilon)(2\beta+\epsilon)}{\epsilon^3(\epsilon-1)(\epsilon-2)},$$

$$\mu_4 = \frac{c^4\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}{\epsilon^4(\epsilon-1)(\epsilon-2)(\epsilon-3)}[\epsilon^3(1+\epsilon) + 6\epsilon^2\{\alpha(\alpha+\epsilon) + \beta(\beta+\epsilon)\} + 3(6+\epsilon)\alpha(\alpha+\epsilon)\beta(\beta+\epsilon)].$$

We will now test the goodness of fit of the Pearson curve $P(x)$ to $F(\alpha, \beta, \gamma, 1)$, where P and F have the same first four moments.

Ia. *Infinite Hypergeometric Series. α, β, γ positive.*

Since we are to fit by the first four moments, we must initially assume $\epsilon > 3$ to ensure all expressions finite.

* *Biometrika*, Vol. XVII. p. 61.

The first two betas of a hypergeometric series in which x is unity, have the following form:

$$\beta_1 = \frac{\epsilon-1}{(\epsilon-2)^2}\frac{(2\alpha+\epsilon)^2(2\beta+\epsilon)^2}{\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}$$

$$= \frac{\epsilon-1}{(\epsilon-2)^2}\left(4+\frac{\epsilon^2}{\alpha(\alpha+\epsilon)}\right)\left(4+\frac{\epsilon^2}{\beta(\beta+\epsilon)}\right),$$

$$\beta_2 = \frac{3(\epsilon-1)(\epsilon+6)}{(\epsilon-2)(\epsilon-3)} + \frac{\epsilon^3(\epsilon-1)}{(\epsilon-2)(\epsilon-3)}\left[\frac{\epsilon(\epsilon+1)}{\alpha(\alpha+\epsilon)\beta(\beta+\epsilon)} + 6\left\{\frac{1}{\alpha(\alpha+\epsilon)} + \frac{1}{\beta(\beta+\epsilon)}\right\}\right].$$

β_1 and β_2 are functions of the three parameters α, β and ϵ, being symmetrical with respect to α and β. As either α or β increases, both β_1 and β_2 decrease. When α and β are large with respect to ϵ, i.e., when γ approaches the value $(\alpha+\beta)$, β_1 and β_2 approach the simple expressions

$$\beta_1 = 16\frac{\epsilon-1}{(\epsilon-2)^2},$$

$$\beta_2 = \frac{3(\epsilon-1)(\epsilon+6)}{(\epsilon-2)(\epsilon-3)}.$$

The criterion K which aids in the determination of the type of a curve, is

$$K = \frac{\beta_1(\beta_2+3)^2}{4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)}.$$

For the above values of β_1 and β_2, K is unity. Hence, these two betas trace out a line on the β_1, β_2 plane which coincides with the Type V line. When ϵ is fairly large, but still small compared with α and β, the distribution represented by the terms of the hypergeometric series tends to become normal.

Substituting the general values of β_1 and β_2 in the expression $2\beta_2-3\beta_1-6$ and simplifying, we have

$$\frac{\epsilon^2(\epsilon-1)}{(\epsilon-2)^2(\epsilon-3)}\left[12\left\{\frac{1}{\epsilon-1}+\frac{1}{\alpha(\alpha+\epsilon)}+\frac{1}{\beta(\beta+\epsilon)}\right\}-\frac{\epsilon(\epsilon^2-7\epsilon+4)}{\alpha(\alpha+\epsilon)\beta(\beta+\epsilon)}\right]$$

$$= \frac{12\epsilon^2}{(\epsilon-2)^2(\epsilon-3)}\frac{1}{\alpha(\alpha+\epsilon)}\frac{1}{\beta(\beta+\epsilon)}\left[\alpha(\alpha+\epsilon)\beta(\beta+\epsilon)\right.$$

$$\left.+(\epsilon-1)\{\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)\}-\frac{\epsilon(\epsilon-1)(\epsilon^2-7\epsilon+4)}{12}\right].$$

The constants α, β and γ have been taken positive, hence the point (β_1, β_2) lies above, below, or on the Type III line $2\beta_2-3\beta_1-6=0$, according as the expression

$$X \equiv \alpha(\alpha+\epsilon)\beta(\beta+\epsilon)+(\epsilon-1)\{\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)\}-\frac{\epsilon(\epsilon-1)(\epsilon^2-7\epsilon+4)}{12}$$

is negative, positive or zero.

The condition for a Type III curve is clearly $X=0$. The algebra may be greatly simplified by writing

$$\frac{1}{\alpha(\alpha+\epsilon)}\frac{1}{\beta(\beta+\epsilon)} = \rho,$$

$$\frac{1}{\alpha(\alpha+\epsilon)}+\frac{1}{\beta(\beta+\epsilon)} = \sigma.$$

The betas then take up the comparatively simple forms

$$\beta_1 = \frac{\epsilon - 1}{(\epsilon - 2)^2}\{16 + 4\epsilon^2\sigma + \epsilon^4\rho\},$$

$$\beta_2 = \frac{3(\epsilon - 1)(\epsilon + 6)}{(\epsilon - 2)(\epsilon - 3)} + \frac{\epsilon^2(\epsilon - 1)}{(\epsilon - 2)(\epsilon - 3)}\{\epsilon(\epsilon + 1)\rho + 6\sigma\}$$

and we find

$$2\beta_2 - 3\beta_1 - 6 = \frac{12\epsilon^2}{(\epsilon - 2)^2(\epsilon - 3)}\left[1 + (\epsilon - 1)\sigma - \frac{\epsilon(\epsilon^2 - 7\epsilon + 4)(\epsilon - 1)\rho}{12}\right],$$

$$4\beta_2 - 3\beta_1 = \frac{\epsilon^2(\epsilon - 1)}{(\epsilon - 2)^2(\epsilon - 3)}[12 + 12(\epsilon - 1)\sigma + \epsilon(\epsilon^3 + 5\epsilon - 8)\rho],$$

$$\beta_2 + 3 = [6 + 6(\epsilon - 1)\sigma + \epsilon^2(\epsilon^2 - 1)\rho]\frac{\epsilon^2}{(\epsilon - 2)(\epsilon - 3)}.$$

Hence, substituting these values in the criterion

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)},$$

we find

$$K = \tfrac{1}{4}\frac{\{16 + 4\epsilon^2\sigma + \epsilon^4\rho\}}{\{12 + 12(\epsilon - 1)\sigma - \epsilon(\epsilon - 1)(\epsilon^2 - 7\epsilon + 4)\rho\}} \times \frac{\{6 + 6(\epsilon - 1)\sigma + \epsilon(\epsilon^2 - 1)\rho\}^2}{\{12 + 12(\epsilon - 1)\sigma + \epsilon(\epsilon^2 + 5\epsilon - 8)\rho\}}.$$

α, β, γ and ϵ are, by hypothesis, positive quantities. Hence, σ, ρ, σ^2 and $\sigma\rho$ are positive quantities. The expression $\epsilon^2 - 7\epsilon + 4$ is a minimum when $\epsilon = \frac{7}{2}$. Its value is then $-\frac{33}{4} > -9$. Hence

$$K > \tfrac{1}{4}\frac{(16 + 4\epsilon^2\sigma + \epsilon^4\rho)}{\{12 + 12(\epsilon - 1)\sigma + 9\epsilon(\epsilon - 1)\rho\}} \times \frac{\{6 + 6(\epsilon - 1)\sigma + \epsilon(\epsilon^2 - 1)\rho\}^2}{\{12 + 12(\epsilon - 1)\sigma + \epsilon(\epsilon^2 + 5\epsilon - 8)\rho\}}.$$

Comparing the coefficients of σ, ρ, $\sigma\rho$, σ^2, ρ^2 in the numerator and denominator of this expression, we find that K is always greater than unity. Hence the two betas of the hypergeometric series $F(\alpha, \beta, \gamma, 1)$ with positive constants, lie above the Type V line which divides the Type IV and Type VI regions on the β_1, β_2 plane.

By making α and β indefinitely large compared with ϵ (which is equivalent to making the quantities ρ and σ indefinitely small) we have already shown that the first two betas tend to lie on the Type V line. Therefore, *the Type V line forms the lower bound* to the β_1 and β_2 area of the above hypergeometric series. This is significant, because it means that when fitting by the first four moments, the Type IV curve never arises.

It now remains to find the upper boundary to the β_1 and β_2 area. This will be found as follows.

The hypergeometric series

$$\left\{1 + \frac{\alpha\beta}{1!\gamma} + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{2!\gamma(\gamma + 1)} + \ldots\right\} y_0,$$

where y_0 is the inverse of the sum of the series, may be written in the form

$$\left[1+\frac{\frac{\alpha\beta}{\gamma}}{1!}+\frac{1}{2!}\left(\frac{\frac{\alpha\beta}{\gamma}\,\frac{(\alpha+1)(\beta+1)}{\gamma}}{1+\frac{1}{\gamma}}\right)+\ldots\right]y_0.$$

Make α, β and γ tend to infinity in such a way that $\frac{\alpha\beta}{\gamma}$ remains constant and equal to m. In the limit the series takes the form

$$\left(1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\ldots\right)e^{-m},$$

which is the exponential or Poisson series. Hence, *the Poisson limit to the binomial is also a limit to the hypergeometric series.*

Now $$\epsilon=\gamma-\alpha-\beta-1,$$

therefore $$\frac{\epsilon}{\alpha\beta}=\frac{\gamma}{\alpha\beta}-\frac{1}{\beta}-\frac{1}{\alpha}-\frac{1}{\alpha\beta}.$$

In the limit, therefore, $$\frac{\epsilon}{\alpha\beta}=\frac{\gamma}{\alpha\beta}=\frac{1}{m}.$$

Hence we may obtain the same Poisson limit by making α, β and ϵ tend to infinity in such a way that $\frac{\alpha\beta}{\epsilon}$ remains equal to a constant quantity m.

We have

$$\beta_1=\frac{\epsilon-1}{(\epsilon-2)^2}\frac{(2\alpha+\epsilon)^2(2\beta+\epsilon)^2}{\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}$$

$$=\frac{(\epsilon-1)\alpha\beta}{(\epsilon-2)^2}\frac{\left(\frac{2}{\alpha}+\frac{\epsilon}{\alpha^2}\right)^2\left(\frac{2}{\beta}+\frac{\epsilon}{\beta^2}\right)^2}{\left(\frac{1}{\alpha}+\frac{\epsilon}{\alpha^2}\right)\left(\frac{1}{\beta}+\frac{\epsilon}{\beta^2}\right)}.$$

and $$\beta_2=\frac{3(\epsilon-1)(\epsilon+6)}{(\epsilon-2)(\epsilon-3)}+\frac{\epsilon^2(\epsilon-1)}{(\epsilon-2)(\epsilon-3)}\left[\frac{\frac{\epsilon}{\alpha\beta}(\epsilon+1)}{\alpha^2\beta^2\left(\frac{1}{\alpha}+\frac{\epsilon}{\alpha^2}\right)\left(\frac{1}{\beta}+\frac{\epsilon}{\beta^2}\right)}\right.$$

$$\left.+6\left\{\frac{1}{\alpha(\alpha+\epsilon)}+\frac{1}{\beta(\beta+\epsilon)}\right\}\right].$$

Proceeding to the limit, β_1 and β_2 take the values

$$\left.\begin{aligned}\beta_1&=\frac{\epsilon}{\alpha\beta}=\frac{1}{m}\\(\beta_2-3)&=\frac{1}{m}\end{aligned}\right\}.$$

Eliminating m, we arrive at the Poisson line

$$\beta_2 - 3 = \beta_1.$$

Substitute now the general values of β_1 and β_2 in $\beta_2 - 3 - \beta_1$, we find

$$\beta_2 - \beta_1 - 3 = \frac{2}{(\epsilon-2)^2(\epsilon-3)}[(7\epsilon^2 - 16\epsilon + 12) + \epsilon^2(\epsilon-1)(\sigma + \sqrt{\epsilon-1}\,\rho)].$$

Since $\epsilon > 3$, this expression is always positive. Hence the points (β_1, β_2) never lie above the Poisson line. They never actually reach this line, which may be considered as a mathematical limit.

Hence, the limits to the β_1, β_2 area for the hypergeometric series in which α, β and γ are positive quantities and x unity, are

$$\begin{cases} \beta_2 - 3 = \beta_1 & \text{upper limit} \\ \beta_1(\beta_2+3)^2 = 4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1) & \text{lower limit.} \end{cases}$$

These two lines arise from the Gaussian point $\left.\begin{matrix} \beta_1 = 0 \\ \beta_2 = 3 \end{matrix}\right\}$.

Examples.

I. $$\alpha = 10, \quad \beta = 30, \quad \gamma = 101,$$

$$\epsilon = \gamma - \alpha - \beta - 1 = 60, \quad c = 1.$$

Substituting these values in the formulae for the moments and betas, we find

$$\log N = 1{\cdot}639{,}8377,$$

$$\nu_2 = 8{\cdot}898{,}305{,}085,$$

$$\beta_1' = \cdot 855{,}217{,}711,$$

$$\beta_2' = 4{\cdot}351{,}447{,}584.$$

Position of the mode measured from start of histogram is

$$\frac{(\alpha-1)(\beta-1)}{(\epsilon+2)} = 4{\cdot}209{,}677.$$

Mean of series measured from start of histogram is

$$\mu_1' = \frac{\alpha\beta}{\epsilon} + \cdot 5 = 5{\cdot}5.$$

Corrected moments and betas, by Sheppard, are

$$\mu_2 = 8{\cdot}814{,}9718,$$

$$\beta_1 = \cdot 879{,}70242,$$

$$\beta_2 = 4{\cdot}377{,}2277.$$

The point (β_1, β_2) falls in the Type VI region, and the curve which has the same first four moments as the above, is

$$y = y_0(x - 164{\cdot}844{,}491)^{4{\cdot}222{,}747}\, x^{-136{\cdot}135{,}056},$$

where $\log y_0 = 301{\cdot}446{,}5204$.

For the curve: mode = 170·121,456,

mean = 171·471,584.

Hence, equation of curve referred to mean is

$$y = y_0 (x + 6{\cdot}627{,}093)^{4{\cdot}222{,}747} (x + 171{\cdot}471{,}584)^{-136{\cdot}135{,}056} \quad \ldots\ldots\ldots(\text{i}).$$

Terms of series	Midordinates of (i)	Areas under (i)
1·000,000	·864,975	·988,594
2·970,298	2·894,099	2·897,940
4·965,056	5·023,947	4·985,139
6·170,167	6·276,503	6·229,540
6·362,985	6·437,399	6·403,479
5·769,107	5·792,867	5·776,574
4·762,233	4·751,426	4·748,863
3·662,279	3·638,780	3·644,017
2·666,172	2·643,873	2·652,174
1·858,982	1·843,221	1·851,678
1·252,279	1·243,150	1·250,392
·820,494	·816,128	·821,744
·525,629	·524,004	·528,083
·330,620	·330,251	·331,081
·204,877	·204,941	·206,833
·125,420	·125,512	·126,744
·076,022	·076,011	·076,828

II. $\alpha = \beta = 30,$

$\epsilon = 50, \quad c = 1.$

Uncorrected moments and betas are

$$\log N = 4{\cdot}974{,}5371,$$
$$\nu_2 = 47{\cdot}020{,}4081,$$
$$\beta_1' = \ \ {\cdot}540{,}5824,$$
$$\beta_2' = 3{\cdot}944{,}4733.$$

Corrected moments and betas are

$$\mu_2 = 46{\cdot}937{,}0748,$$
$$\beta_1 = \ \ {\cdot}543{,}46676,$$
$$\beta_2 = 3{\cdot}947{,}83375,$$

and the Pearson curve with the same first four moments is

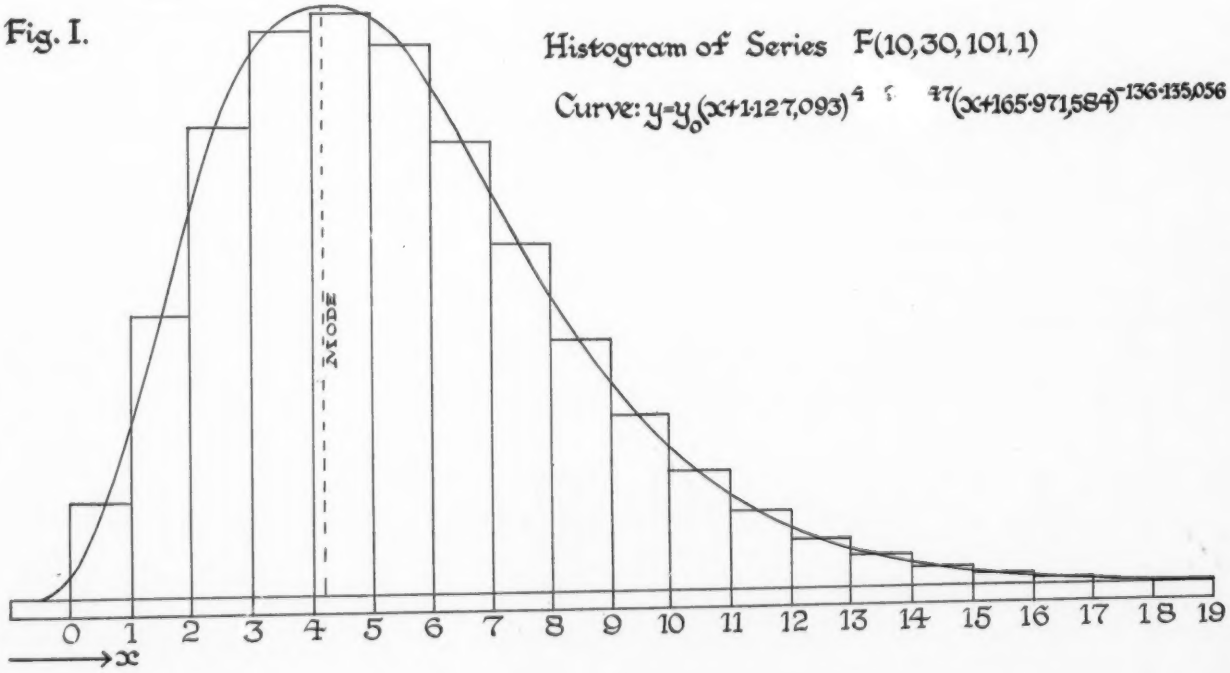
$$y = y_0 (x - 86{\cdot}467{,}3846)^{13{\cdot}408{,}4683}\, x^{-69{\cdot}792{,}1269},$$

$$\log y_0 = 127{\cdot}806{,}2761.$$

Mode of series measured from start of histogram = 16·173,077.

Mean of series measured from start of histogram = 18·5.

Fig. I.


Histogram of Series F(10,30,101,1)
Curve: y=y0(x+1·127,093)^4 (47(x+165·971,584)^-136·135,056
MODE
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
x

For the curve: mode = 107·002,995,

mean = 109·376,152.

Hence, equation of curve referred to mean as origin is

$$y = y_0 (x + 22{\cdot}908{,}7671)^{13{\cdot}408{,}4683} (x + 109{\cdot}376{,}1517)^{-69{\cdot}792{,}1269}.$$

Terms of series	Midordinates of (i)	Areas under (i)	Terms of series	Midordinates of (i)	Areas under (i)
1·000	1·661	2·143	2219·294	2216·193	2217·745
8·108	9·338	10·122	1881·510	1878·842	1880·440
34·785	35·842	37·594	1582·046	1579·832	1581·402
105·074	104·401	107·483	1320·269	1318·493	1319·981
250·993	246·924	251·384	1094·251	1092·860	1094·233
504·485	496·498	501·947	901·236	900·190	901·428
887·922	876·844	882·507	738·007	737·243	738·340
1405·064	1393·196	1398·031	601·165	600·631	601·587
2037·641	2027·573	2030·881	487·341	486·964	487·788
2747·294	2741·426	2742·532	393·330	393·072	393·773
3482·197	3481·828	3480·533	316·176	316·006	316·596
4185·962	4191·145	4187·632	253·220	253·098	253·591
4806·421	4816·147	4810·885	202·116	202·003	202·441
5302·393	5314·851	5308·492	160·831	160·768	161·104
5647·539	5660·936	5654·136	127·619	127·564	127·839
5831·270	5843·824	5837·208	101·007	100·957	101·180
5857·304	5867·936	5861·957	79·758	79·708	79·889
5740·646	5748·560	5743·552	62·846	62·796	62·941
5503·945	5508·994	5505·110	49·426	49·375	49·492
5173·842	5176·205	5173·474	38·804	38·751	38·844
4777·844	4777·864	4776·227	30·417	30·365	30·439
4341·916	4340·233	4339·546	23·809	23·758	23·817
3888·884	3886·114	3886·210	18·613	18·564	18·610
3437·577	3434·306	3434·994	14·535	14·488	14·525
3002·535	2999·010	3000·138	.....	.....	.....
2594·189	2590·791	2592·192			

III. $\alpha = \beta = 94, \quad c = 1,$

$\gamma = 589, \quad \epsilon = 400.$

Raw moments and betas are

$\log N = 7{\cdot}807{,}6359,$

$\nu_2 = 33{\cdot}776{,}6619,$

$\beta_1' = {\cdot}193{,}6389.$

$\beta_2' = 3{\cdot}210{,}1843.$

Mode of series = 21·514,925,

mean of series = 22·59,

both being measured from the start of the histogram.

The quantity $r = \dfrac{6(\beta_2 - \beta_1 - 1)}{2\beta_2 - 3\beta_1 - 6}$ is very large, of the order 6000. We are justified, therefore, in fitting a Type III curve.

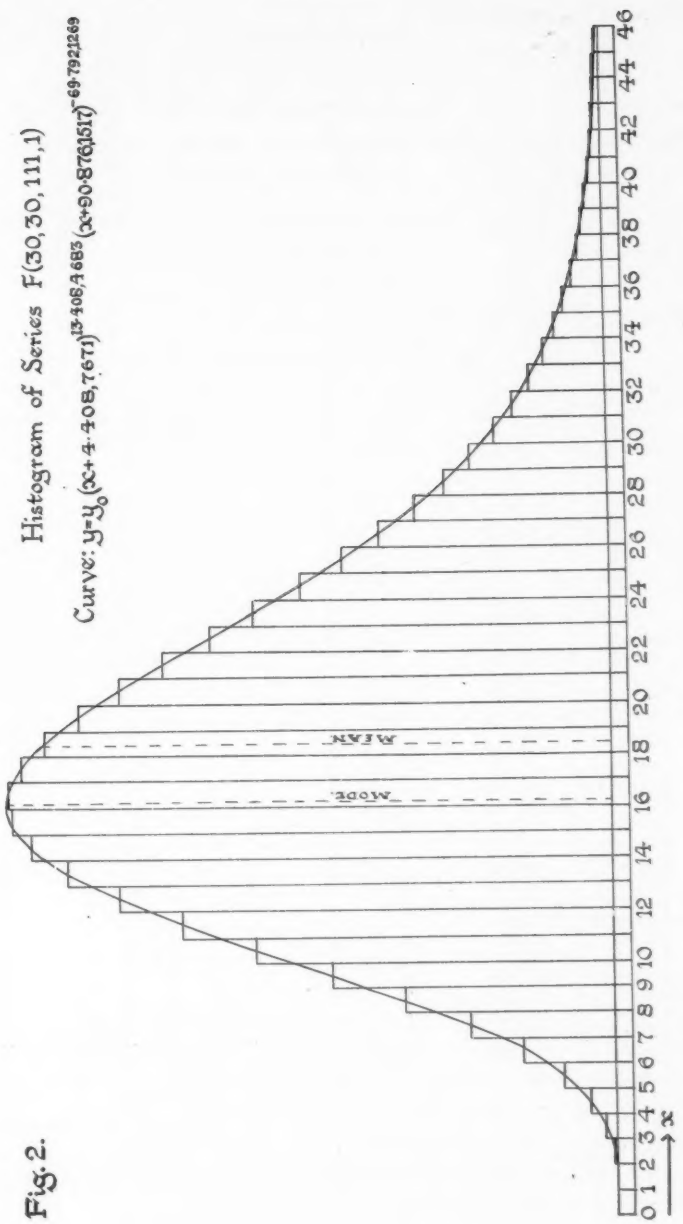

Histogram of Series F(30,30,111,1)
Curve: $y=y_0(x+4.408,7671)^{13.408,4683}(x+90.876151i)^{-69.7921269}$
MODE
MEAN
0 1 2 3 4 5 6 7 8 9 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46
x

Fig. 2.

Corrected moments are

$$\mu_2 = 33{\cdot}693{,}3286,$$

$$\mu_3 = 73{\cdot}354{,}7625,$$

whence $\beta_1 = \cdot 140{,}67776.$

The Pearson-type III curve having the same first three moments is

$$y = y_0 e^{-\cdot 918{,}6405x}\, x^{27{\cdot}433{,}8136},$$

$$\log y_0 = \overline{22}{\cdot}096{,}8526.$$

For the curve: mode $= 29{\cdot}863{,}491,$

mean $= 30{\cdot}952{,}057.$

Equation of curve referred to mean as origin is

$$y = y_0' e^{-\cdot 918{,}6405x}\, (x + 30{\cdot}952{,}057)^{27{\cdot}433{,}8136}.$$

Significant terms of series $\times 10^{-1}$	Corresponding midordinates of curve $\times 10^{-1}$	Areas under curve $\times 10^{-1}$	Significant terms of series $\times 10^{-1}$	Corresponding midordinates of curve $\times 10^{-1}$	Areas under curve $\times 10^{-1}$
2	3	3	126,924	126,743	126,931
11	15	17	99,959	99,806	99,993
60	68	73	77,439	77,316	77,488
237	250	264	59,060	58,962	59,116
768	778	811	44,377	42,298	44,430
2,111	2,099	2,165	32,874	32,810	32,920
5,068	5,000	5,116	24,025	23,972	24,062
10,843	10,676	10,857	17,332	17,288	17,359
20,996	20,694	20,948	12,350	12,313	12,368
37,249	36,805	37,125	8,596	8,665	8,707
61,144	60,594	60,955	6,055	6,028	6,060
93,627	93,049	93,409	4,171	4,148	4,171
134,647	134,150	134,456	2,843	2,824	2,841
182,911	182,604	182,803	1,919	1,903	1,915
235,873	235,839	235,888	1,283	1,270	1,278
289,984	290,254	290,130	850	839	845
341,158	341,704	341,411	558	550	554
385,351	386,095	385,657	363	357	360
419,131	419,967	419,430	235	230	232
440,122	440,939	440,357	151	147	149
447,246	447,954	447,384	96	93	94
440,748	441,282	440,772	61	58	59
422,024	422,361	421,946	38	36	37
393,320	393,471	393,170	24	23	24
357,363	357,354	357,172	15	14	15
317,002	316,879	316,809	9	9	9
274,907	274,714	274,738	6	5	6
233,355	233,131	233,229	3	3	3
194,112	193,888	194,036	...	...	...
158,399	158,190	158,367			

With the exception of the first few terms we see that the areas under the curve accord very closely with the terms of the series when the standard deviation of the latter is not small, or what is equivalent, when the number of significant terms is not small. (See also pp. 302—304.)

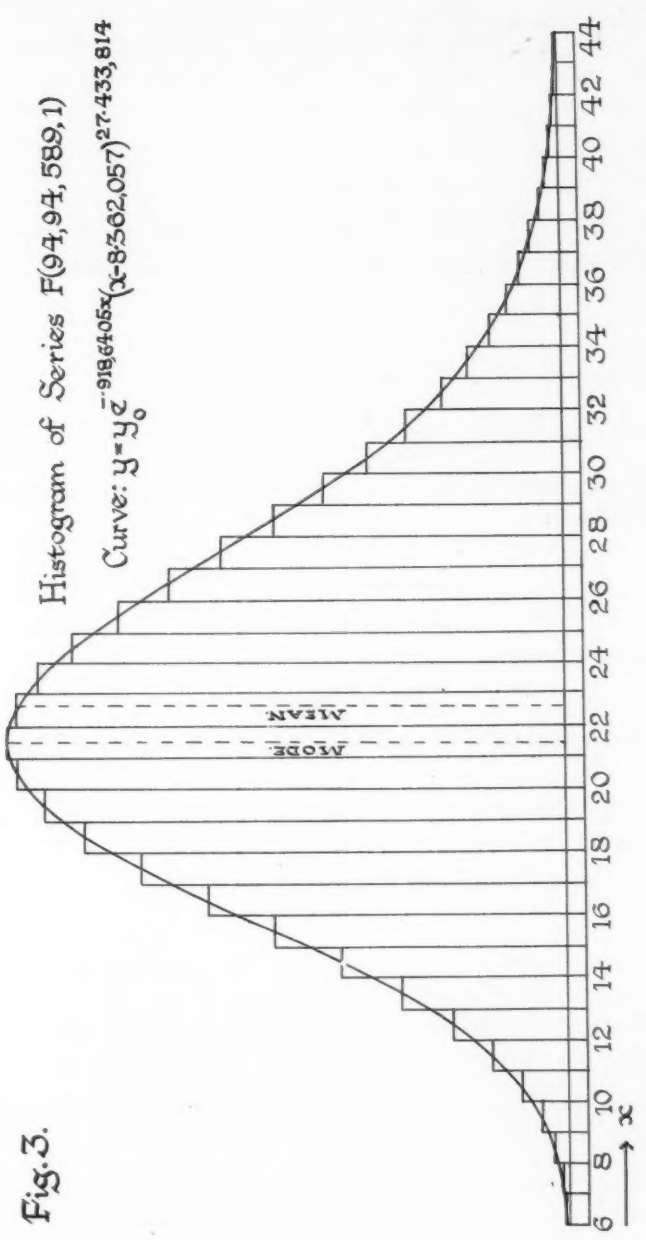

Histogram of Series F(94,94,589,1)
Curve: $y = y_0 e^{-.918,6405x}(x-8.562,057)^{27.433,814}$
MODE
MEAN
6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44
x

Fig. 3.

I*b*. *Infinite Hypergeometric Series in which two of* α, β, γ *are Negative Non-integral Numbers.*

(i) α and β negative, γ positive.

Let a be the largest integer in $|\alpha|$ and b the largest integer in $|\beta|$, then

$$\alpha = -a - \delta_1,$$

$$\beta = -b - \delta_2 \qquad 0 < \delta_1, \delta_2 < 1.$$

Assume that $b \geqslant a$, i.e. $\beta \leqslant \alpha$. The $(a+1)$st term of the series is

$$\frac{\alpha(\alpha+1)(\alpha+2)\dots(-\delta_1)\beta(\beta+1)\dots(\beta+a)}{(a+1)!\,\gamma(\gamma+1)\dots(\gamma+a)}.$$

This term is positive. The $(a+2)$nd term is

$$\frac{\alpha(\alpha+1)\dots(-\delta_1)(1-\delta_1)\beta(\beta+1)\dots(\beta+a)(\beta+a+1)}{(a+2)!\,\gamma(\gamma+1)\dots(\gamma+a+1)}.$$

In order that this and all subsequent terms be positive, the following condition must be satisfied:

$$\beta + a + 1 > 0.$$

Now $$\alpha = -a - \delta_1 \qquad 0 < \delta_1 < 1,$$

$$\beta = -a - \delta_2' \qquad \delta_2' \geqslant \delta_1.$$

Therefore $$\beta + a + 1 = (1 - \delta_2') > 0,$$

i.e. $$1 > \delta_2' > 0.$$

A necessary and sufficient condition for all terms to be positive is, therefore,

$$a = b.$$

Since the series is infinite, the condition $\epsilon > 3$ must be satisfied if we intend fitting curves by the first four moments.

When $\epsilon \geqslant 4$, the first two betas lie above the Type III line. When also

$$-(\epsilon - 1) \geqslant \frac{\alpha}{\beta} \geqslant -1,$$

the first two betas lie above the Poisson line. In either case the corresponding Pearson curve is of Type I (or the types associated with Type I). (For proof of this statement, see corresponding section under finite series, pp. 298—9.)

(ii) α and γ negative, β positive.

As in the previous case, in order that the terms of the series be all positive, the condition $|\gamma - \alpha| < 1$ must be satisfied.

Now $$\epsilon = \gamma - \alpha - \beta - 1 \quad > 3,$$

$$< -\beta.$$

Hence, for all expressions to be finite, $(-\beta)$ must be at least greater than 3. This implies $\beta < -3$, contrary to the hypothesis that β is positive. A positive ϵ would imply a negative β, hence we cannot have an infinite convergent hypergeometric series with all terms positive and γ negative.

Example.

$$\left.\begin{aligned} \alpha = \beta &= -25{\cdot}6 \\ \gamma &= 30, \quad c = 1 \end{aligned}\right\}.$$

Crude moments and betas are

$$\log N = 5{\cdot}369{,}3566,$$
$$\nu_2 = 3{\cdot}835{,}2290,$$
$$\beta_1' = \cdot 004{,}688549,$$
$$\beta_2' = 2{\cdot}949{,}4076.$$
$$\text{Mean of series} = 8{\cdot}671{,}571.$$
$$\text{Mode of series} = 8{\cdot}607{,}786.$$

Corrected moments and betas are

$$\mu_2 = 3{\cdot}751{,}8956,$$
$$\beta_1 = \cdot 005{,}007951,$$
$$\beta_2 = 2{\cdot}947{,}7272,$$

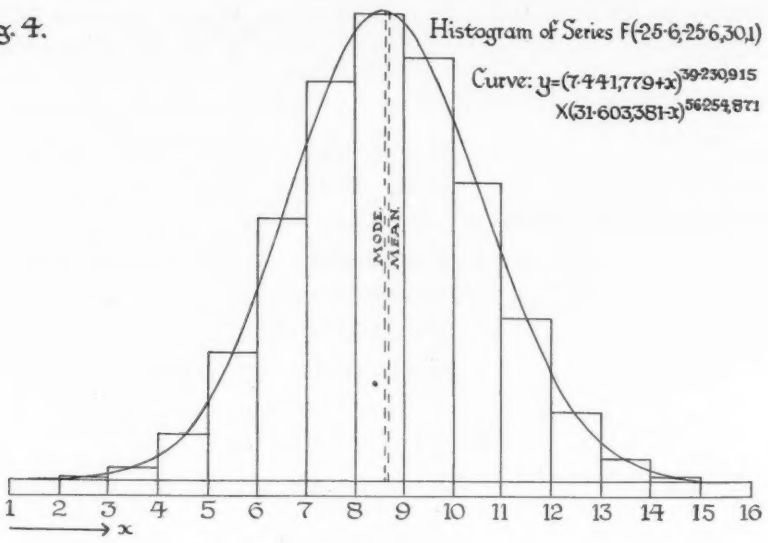
and the Pearson curve having the same first four moments is

$$y = y_0 (16{\cdot}041{,}94 + x)^{39{\cdot}230{,}915} (23{\cdot}00322 - x)^{56{\cdot}254{,}871},$$
$$\log y_0 = \overline{120}{\cdot}789{,}4740.$$

Origin at mode. Mean-mode of curve = ·0714,082.

Terms of series	Midordinates of curve	Areas under curve
1·0	1·2	1·6
21·8	19·7	24·5
213·2	188·3	216·6
1,237·1	1,131·3	1,232·5
4,786·7	4,554·5	4,764·9
13,136·9	12,886·0	13,122·7
26,546·5	26,538·9	26,575·1
40,468·6	40,829·6	40,514·2
47,299·1	47,808·2	47,300·7
42,840·2	43,154·0	42,799·3
30,269·4	30,271·0	30,273·0
16,741·7	16,561·8	16,750·5
7,253·4	7,064·9	7,270·7
2,457·1	2,339·8	2,463·8
648·0	596·6	646·6
132·1	115·6	129·8
20·6	16·7	19·6
2·4	1·8	2·2

Fig. 4.



Here again the accordance between the terms of the series and the areas under the curve is very close in the significant parts of the curve. (See also pp. 302—4 and 306.)

II. *Finite Hypergeometric Series.*

The hypergeometric series

$$1+\frac{\alpha\beta}{1!\,\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)}+\ldots$$

is finite only when α or β or both are negative integers. In order that the series may represent a frequency distribution, each term must be positive. Hence, either

(a) α and β are negative with one at least an integer, or

(b) α and γ negative, α a negative integer and $|\gamma|>|\alpha|$.

These two sets of conditions give rise to two distinct types of series which will be considered separately.

(a) *α and β negative.*

In order to find the position of the point (β_1, β_2) for such series, relative to the Poisson line, substitute the expressions for the β's in $\beta_2-\beta_1-3\equiv\theta$. We then have

$$\theta=\frac{2}{(\epsilon-2)^2(\epsilon-3)}\,\frac{1}{\alpha(\alpha+\epsilon)}\,\frac{1}{\beta(\beta+\epsilon)}[(7\epsilon^2-16\epsilon+12)\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)$$
$$+\epsilon^3(\epsilon-1)\{\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)\}+\epsilon^3(\epsilon-1)^2].$$

If $\epsilon > 3$, the coefficient $\frac{2}{(\epsilon-2)^2(\epsilon-3)} \cdot \frac{1}{\alpha(\alpha+\epsilon)} \cdot \frac{1}{\beta(\beta+\epsilon)}$ is positive. Hence, the sign of $\beta_2 - \beta_1 - 3$ is the same as that of the expression

$$E = \alpha(\alpha+\epsilon)\beta(\beta+\epsilon)\{7\epsilon^2 - 16\epsilon + 12\} + \epsilon^3(\epsilon-1)\{\alpha(\alpha+\epsilon) + \beta(\beta+\epsilon)\} + \epsilon^3(\epsilon-1)^2.$$

E is negative when $\alpha = \beta = -1$ and also when $\alpha = \beta = -(\epsilon-1)$ for all permissible values of ϵ. When either α or β is greater than -1, E may be positive, in which case the point (β_1, β_2) will lie below the Poisson line. We will now show that for $\epsilon \geqslant 4$, E is negative when α and β lie in the range $(-1, -\epsilon+1)$. It will be sufficient to prove that under these conditions E is negative at all its points of maxima and minima.

Differentiate E with respect to α and β respectively.

$$\frac{\partial E}{\partial \alpha} = (2\alpha+\epsilon)[\epsilon^3(\epsilon-1) + \beta(\beta+\epsilon)(7\epsilon^2 - 16\epsilon + 12)],$$

$$\frac{\partial E}{\partial \beta} = (2\beta+\epsilon)[\epsilon^3(\epsilon-1) + \alpha(\alpha+\epsilon)(7\epsilon^2 - 16\epsilon + 12)].$$

Clearly, the points of maxima and minima are given by

(i) $\beta(\beta+\epsilon) = \alpha(\alpha+\epsilon) = -\epsilon^3(\epsilon-1)/(7\epsilon^2 - 16\epsilon + 12)$,

(ii) $\alpha = \beta = -\frac{\epsilon}{2}$.

At point (i), E has the value

$$\begin{aligned}\alpha(\alpha+\epsilon)[\epsilon^2(\epsilon+1) + \beta(\beta+\epsilon)(7\epsilon^2 - 16\epsilon + 12)] \\ + \epsilon^3(\epsilon-1)^2 + \beta(\beta+\epsilon)\epsilon^3(\epsilon-1)\end{aligned}$$

$$= \epsilon^3(\epsilon-1)[(\epsilon-1) + \beta(\beta+\epsilon)]$$

$$= -\frac{\epsilon^3(\epsilon-1)^2(\epsilon-2)^2(\epsilon-3)}{(7\epsilon^2 - 16\epsilon + 12)}.$$

This is negative for all values of $\epsilon > 3$.

At point (ii), E has the value $-\frac{\epsilon^3}{16}(\epsilon-4)(\epsilon-2)^2$. This is again negative for $\epsilon \geqslant 4$. It follows, therefore, that E is negative when $\epsilon \geqslant 4$ and $-1 \geqslant \begin{Bmatrix}\alpha \\ \beta\end{Bmatrix} \geqslant -(\epsilon-1)$. The corresponding betas then lie above the Poisson line.

When $\epsilon = 4$, then $(\alpha+\beta) > -5$, in which case the values of α and β which result in a series with the largest number of terms are, respectively, $-2, -3$. This is a series with three terms. Hence, for finite series with a minimum of three terms, the first two betas lie above the Poisson line.

As previously (p. 300), if $-\alpha$, $-\beta$ and γ tend to infinity in such a way that $\frac{\alpha\beta}{\gamma}$ remains constant and equal to a finite quantity m, the finite hypergeometric series will, in the limit, become a Poisson series.

When α and β are fractional, the series is infinite (see p. 308). α and β may then be greater than -1 and the corresponding betas may lie below the Poisson line. However, by substituting in $2\beta_2 - 3\beta_1 - 6$ it can be readily shown that for $\epsilon \geqslant 4$ and for any negative values of α and β the first two betas lie above the Type III line.

The series (2) (p. 295) is finite. α and β are negative integers. Hence, for $\epsilon = N \geqslant 4$, its first two betas lie above the Poisson line.

(*b*) *Second type of finite series.*

α and γ negative, α an integer; β positive and $|\gamma| > |\alpha|$.

The last condition is introduced to ensure that all terms are positive.

Now $$\gamma < \alpha, \quad \gamma - \alpha - \beta - 1 < -\beta - 1,$$

i.e. $$\epsilon < -(\beta + 1).$$

β is positive, hence ϵ is negative.

The position of the first two betas of such series relative to the Poisson line is determined from the sign of the expression

$$\frac{2}{(\epsilon-2)^2(\epsilon-3)} \frac{1}{\alpha(\alpha+\epsilon)} \frac{1}{\beta(\beta+\epsilon)} [(7\epsilon^2 - 16\epsilon + 12)\alpha\beta(\alpha+\epsilon)(\beta+\epsilon) + \epsilon^3(\epsilon-1)[\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)] + \epsilon^3(\epsilon-1)^2].$$

The term outside the square brackets is always positive. It is sufficient, therefore, to consider the sign of the expression inside the brackets. This becomes, after substituting for α, ϵ, $(\epsilon+\beta)$ etc., their absolute values $-\alpha'$, $-\epsilon'$, $-(\epsilon'-\beta)$ etc.,

$$-(7\epsilon'^2 + 16\epsilon' + 12)\alpha'(\alpha'+\epsilon')\beta(\epsilon'-\beta) + \epsilon'^3(\epsilon'+1)\{(\epsilon'+\alpha')\alpha' - (\epsilon'-\beta)\beta\} - \epsilon'^3(\epsilon'+1)^2.$$

This expression may be positive, negative or zero. The first two betas may, therefore, lie below, above or on the Poisson line.

Rearrange in the form

$$\alpha'(\alpha'+\epsilon')[\epsilon'^3(\epsilon'+1) - \beta(\epsilon'-\beta)(7\epsilon'^2+16\epsilon'+12)] - \epsilon'^3(\epsilon'+1)[(\epsilon'+1)+\beta(\epsilon'-\beta)].$$

α' may vary independently of ϵ'. Hence, this expression will be negative for all values of α' only when

$$\epsilon'^3(\epsilon'+1) - \beta(\epsilon'-\beta)(7\epsilon'^2+16\epsilon'+12) \leqslant 0,$$

i.e. $$\beta^2 - \beta\epsilon' + \frac{\epsilon'^3(\epsilon'+1)}{7\epsilon'^2+16\epsilon'+12} \leqslant 0,$$

i.e. when β lies between the values

$$\frac{\epsilon'}{2}\left[1 \mp (\epsilon'+2)\sqrt{\frac{3}{7\epsilon'^2+16\epsilon'+12}}\right].$$

When β lies outside these limits, the first two betas of the corresponding series may lie above the Poisson line.

However, by substituting in $2\beta_2-3\beta_1-6$, we can show that for all permissible values of β, the corresponding point (β_1, β_2) lies above the Type III line. For the position of this point relative to the Type III line depends on the sign of the expression

$$[\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)](\epsilon-1)+\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)-\frac{\epsilon(\epsilon-1)}{12}(\epsilon^2-7\epsilon+4).$$

Putting in for the constants their absolute values, this expression may be rewritten in the form

$$-(\epsilon'+1)\{\alpha'(\alpha'+\epsilon')-\beta(\epsilon'-\beta)\}-\alpha'\beta(\epsilon'-\beta)(\alpha'+\epsilon')-\frac{\epsilon'(\epsilon'+1)(\epsilon'^2+7\epsilon'+4)}{12}.$$

Rearranging the terms,

$$\beta(\epsilon'-\beta)[(\epsilon'+1)-\alpha'(\alpha'+\epsilon')]-\alpha'(\alpha'+\epsilon')(\epsilon'+1)-\frac{\epsilon'(\epsilon'+1)}{12}(\epsilon'^2+7\epsilon'+4) \quad \text{(i)}.$$

This can be positive only when $(\epsilon'+1)>\alpha'(\alpha'+\epsilon')$.

For such values, the maximum is reached when $\beta=\frac{\epsilon'}{2}$. Substitute, therefore, $\beta=\frac{\epsilon'}{2}$ in (i). We have

$$-\alpha'(\alpha'+\epsilon')\left[(\epsilon'+1)+\frac{\epsilon'^2}{4}\right]+\frac{(\epsilon'+1)\epsilon'^2}{4}-\frac{\epsilon'(\epsilon'+1)(\epsilon'^2+7\epsilon'+4)}{12} \quad \text{...(ii)}.$$

This is evidently negative for all permissible values of ϵ'. Hence, the corresponding betas always lie above the Type III line. The series referred to in paragraphs (i) and (ii) (p. 296) and Examples II (p. 315) and IV (p. 318) are illustrations of finite series with negative γ.

When $\beta=1$, the hypergeometric series reduces to

$$1+\frac{\alpha}{\gamma}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}+\ldots.$$

This is finite when α is a negative integer. The terms are all positive and finite when γ is also negative and $|\gamma|>|\alpha|$. Substitute for α and γ their absolute values, the series then adopts the form

$$1+\frac{\alpha'}{\gamma'}+\frac{\alpha'(\alpha'-1)}{\gamma'(\gamma'-1)}+\frac{\alpha'(\alpha'-1)(\alpha'-2)}{\gamma'(\gamma'-1)(\gamma'-2)}+\ldots \quad \text{.....(iii)}.$$

When $\gamma'\to\alpha'$, the distribution represented by the terms of the series tends to become rectangular. For all other values the distribution is J-shaped.

In order that a hypergeometric series may represent a U-shaped distribution, the following conditions must be satisfied:

(i) Series finite.

(ii) $\gamma>\alpha\beta$.

(iii) There must exist an antimode, i.e. from some point onwards the terms of the series must be constantly increasing, i.e.

$$\frac{(\alpha-r)(\beta-r)}{(r+1)(\gamma+r)}>1 \quad \text{for} \quad \begin{cases} r>r_0. \\ r_0<|\alpha|. \end{cases}$$

Case (*a*). α and β negative.

$$\frac{(\alpha+r)(\beta+r)}{(r+1)(\gamma+r)} = \frac{(|\alpha|-r)(|\beta|-r)}{(r+1)(\gamma+r)} < \frac{\alpha\beta}{\gamma}.$$

Hence, if initially we have $\gamma > \alpha\beta$, the terms of the series will form a monotone decreasing sequence and in no case can they represent a U-shaped distribution.

Case (*b*). β positive, α and γ negative.

For a U-shaped distribution, we must initially have $\gamma > \alpha\beta$. There must also exist a positive r for which

$$\frac{(\beta+r)(\alpha'-r)}{(r+1)(\gamma'-r)} > 1 \quad r < \alpha' \qquad \text{.....(iv)},$$

where α' and γ' are the absolute values of the constants. (iv) is equivalent to

$$r > \frac{\gamma' - \alpha'\beta}{\alpha' - \beta - \gamma' + 1} = r_0 < \alpha'.$$

r_0 must be less than α', i.e.

$$\gamma' < \alpha'^2 - \alpha'\gamma' + \alpha' \text{ or } \gamma'(1+\alpha') < \alpha'(1+\alpha') \text{ or } \gamma' < \alpha',$$

contrary to the hypothesis that γ' is greater than α'.

In no case, therefore, can a hypergeometric series with $x = 1$ represent a U-shaped distribution. Since the upper branch of the biquadratic on the β_1, β_2 plane forms the lower bound to Pearson U-shaped curves, it also forms the upper limit to the first two betas of hypergeometric series in which the fourth element x is unity.

Examples.

I. $\alpha = -30$, $\beta = -50$,
$\gamma = 100$, $c = 1$.

The variance and betas of the series after applying Sheppard's corrections are

$$\mu_2 = 4{\cdot}971{,}8996,$$
$$\beta_1 = {\cdot}018{,}306731,$$
$$\beta_2 = 2{\cdot}956{,}3878.$$

The sum of the series is given by

$$\log N = 4{\cdot}740{,}1506.$$

Mean and mode of series, measured from the start of the histogram, are respectively,

$$8{\cdot}879{,}8883;\ 8{\cdot}734{,}8066.$$

The Pearson-type curve having the same first four moments as the series is

$$y = y_0 x^{27{\cdot}737{,}858}(42{\cdot}504{,}879 - x)^{52{\cdot}069{,}623} \qquad \text{.....(i)},$$

$$\log y_0 = \overline{104}{\cdot}415{,}7762.$$

Mode of curve $= 14{\cdot}772{,}980.$

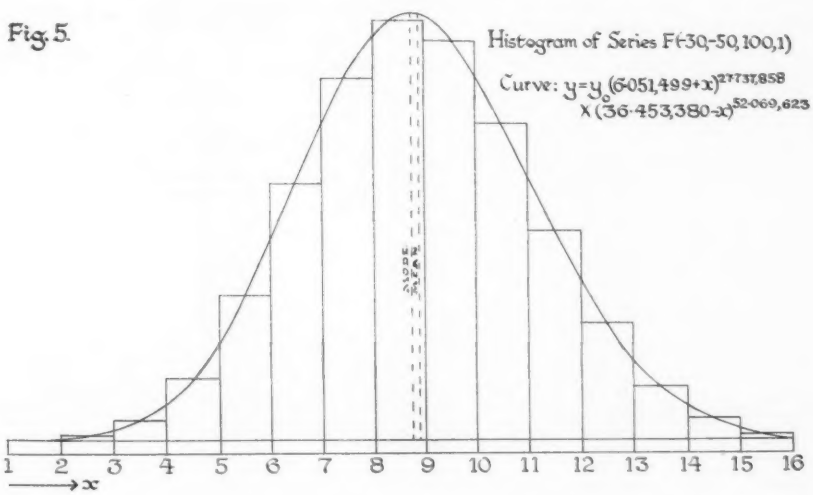
Mean of curve $= 14{\cdot}931{,}387.$

Equation of curve referred to the start of the histogram as origin is

$$y = y_0(6{\cdot}051{,}499 + x)^{27{\cdot}737{,}858}(36{\cdot}453{,}380 - x)^{52{\cdot}069{,}623}.$$

Terms of series	Corresponding midordinates of (i)	Corresponding areas under (i)
1·00	1·17	1·50
15·00	13·81	16·29
105·52	195·89	105·90
463·46	434·52	459·18
1427·50	1382·39	1419·70
3283·25	3253·92	3281·78
5862·95	5887·76	5873·85
8344·04	8424·94	8357·10
9640·48	9736·84	9640·82
9164·40	9231·93	9152·52
7239·04	7262·68	7241·46
4786·14	4778·35	4785·49
2662·56	2641·87	2668·52
1250·82	1230·47	1256·37
497·32	482·74	499·43
167·52	159·16	167·28
47·80	43·90	46·99
11·54	10·06	11·00
2·35	1·90	2·12

Fig. 5.



II. $$\alpha = -30\,;\ \beta = 60,$$
$$\gamma = -81\,;\ c = 1.$$

Corrected moments and betas are:

$$\mu_2 = 9{\cdot}293{,}3598,$$
$$\beta_1 = {\cdot}00127{,}2249,$$
$$\beta_2 = 2{\cdot}895{,}9895,$$
$$\log N = 8{\cdot}459{,}5208.$$

Mean and mode of series measured from the start of the histogram are, respectively,

$$16{\cdot}571{,}4286\,;\ 16{\cdot}627{,}2727.$$

The Pearson-type curve which has the same first four moments as the series is

$$y = y_0 x^{27{\cdot}649{,}947}(45{\cdot}180{,}196 - x)^{24{\cdot}035{,}187} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

$$\log y_0 = \overline{63}{\cdot}565{,}6609.$$

$$\text{Mean of curve} = 24{\cdot}103{,}296.$$

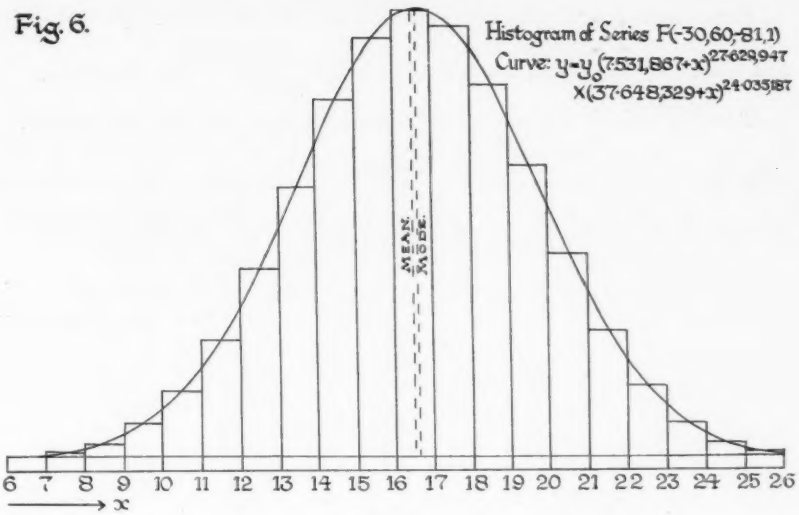
$$\text{Mode of curve} = 24{\cdot}161{,}973.$$

Equation of curve referred to the start of the histogram as origin is

$$y = y_0(7{\cdot}531{,}869 + x)^{27{\cdot}649{,}947}(37{\cdot}648{,}327 - x)^{24{\cdot}035{,}187}.$$

Significant terms of series $\times 10^{-2}$	Corresponding midordinates of (i) $\times 10^{-2}$	Corresponding areas under curve $\times 10^{-2}$
2·5	2·4	2·8
18·0	16·9	19·0
98	91	99
424	396	424
1,511	1,430	1,506
4,560	4,374	4,545
11,869	11,523	11,847
27,025	26,499	27,012
54,388	53,738	54,410
97,495	96,871	97,562
156,573	156,191	156,658
226,219	226,284	226,259
294,892	295,519	294,839
347,416	348,551	347,278
370,116	371,542	369,980
356,383	357,754	356,339
309,670	310,641	309,847
242,148	242,492	242,294
169,699	169,420	169,811
105,979	105,291	106,002
58,829	57,739	58,480
28,294	27,647	28,228
11,810	11,405	11,770
4,177	3,983	4,170
1,219	1,152	1,229
282	268	293
48·7	48·2	55·0
5·6	6·4	7·6

Fig. 6.



III. $\alpha = -100,\ \beta = -100,$

$\epsilon = 200,\ c = 1.$

This series is symmetrical. Its corrected moments and betas are

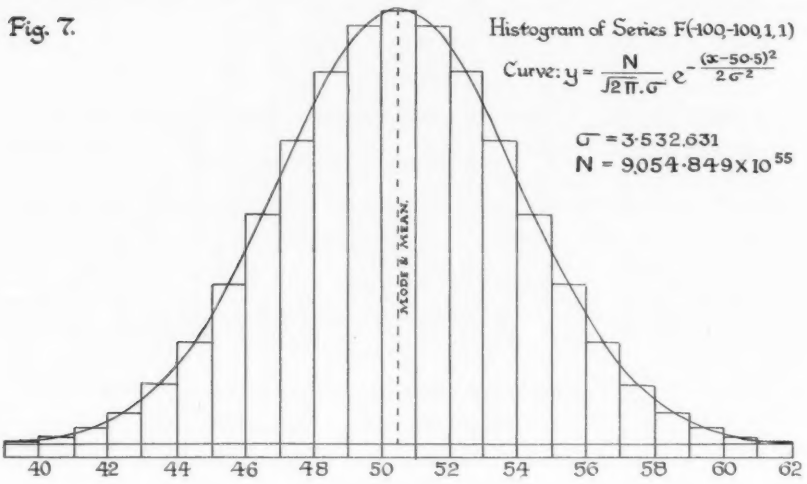
$$\mu_2 = 12{\cdot}479{,}4807,$$
$$\beta_1 = 0,$$
$$\beta_2 = 2{\cdot}989{,}9710,$$
$$N = 9054{\cdot}849 \times 10^{55}.$$

Significant terms of series $\times 10^{-55}$	Corresponding midordinates of normal curve $\times 10^{-55}$	Corresponding areas $\times 10^{-55}$
1017·906	1022·570	1018·286
979·380	982·400	979·825
868·746	871·141	859·173
712·564	713·000	712·344
539·799	538·627	539·081
377·591	375·569	376·852
243·821	241·705	243·237
145·287	143·577	144·966
79·856	78·721	79·806
40·467	39·837	40·570
18·896	18·607	19·042
8·126	8·022	8·256
3·215	3·192	3·305
1·170	1·172	1·256
·391	·397	·418
·120	·124	·130
·034	·036	·039
·009	·009	·010
·002	·002	·003

Mean = mode = 50·5, measured from the start of the histogram. β_2 is sufficiently near 3 to justify our fitting the normal curve $y = \frac{N}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$, $\sigma = 3{\cdot}532{,}631$.

Since the series is symmetrical, the terms after the mode only will be given.

Fig. 7.



IV. $\alpha = 1$, $\beta = -60$,

$\gamma = -65$, $c = 1$.

The hypergeometric series having the above values for its constants is very abrupt, the maximum term being the first. It is necessary, therefore, to apply abruptness corrections to the moments. The corrected moments about the start of the histogram are

$\mu_1' = 9{\cdot}063{,}5495$,

$\mu_2' = 143{\cdot}738{,}0331$,

$\mu_3' = 3038{\cdot}952{,}2009$,

and $N = 11$.

The best fit is obtained by fixing the start and fitting a Type I curve by equating the first three moments about the stump. This curve is found to be

$$y = y_0 x^{-\cdot 000{,}1138} (63{\cdot}349{,}729 - x)^{4{\cdot}988{,}8238} \quad \text{.....(i).}$$

This is sufficiently close to the curve

$$y = y_0' (63{\cdot}349{,}729 - x)^{4{\cdot}988{,}8238} \quad \text{.....(ii),}$$

$$\log y_0 = \bar{9}{\cdot}028{,}4019$$

for practical purposes.

The area up to a point s is given by the following relation:

$$Na_s = y_0 \int_0^s (b-x)^p \, dx = \frac{y_0}{p+1} [b^{p+1} - (b-s)^{p+1}].$$

Terms of series	Midordinates of curve	Areas under curve	Terms of series	Midordinates of curve	Areas under curve
1·000,000	·999,587	·999,980	·033,688	·033,661	·033,690
·923,077	·922,722	·922,922	·028,734	·028,709	·028,733
·850,962	·850,655	·850,855	·024,380	·024,357	·024,381
·783,425	·783,163	·783,344	·020,571	·020,549	·020,569
·720,246	·720,021	·720,194	·017,253	·017,233	·017,252
·661,209	·661,018	·661,183	·014,377	·014,359	·014,376
·606,108	·605,944	·606,097	·011,898	·011,882	·011,897
·554,743	·554,603	·554,748	·009,774	·009,759	·009,773
·506,921	·506,800	·506,939	·007,964	·007,951	·007,962
·462,454	·462,350	·462,479	·006,432	·006,421	·006,432
·421,163	·421,074	·421,200	·005,146	·005,136	·005,145
·383,876	·382,799	·382,914	·004,074	·004,066	·004,074
·347,424	·347,357	·347,472	·003,188	·003,182	·003,188
·314,648	·314,589	·314,690	·002,464	·002,457	·002,464
·284,394	·284,340	·284,442	·001,877	·001,873	·001,877
·256,512	·256,463	·256,557	·001,408	·001,405	·001,409
·230,861	·230,816	·230,900	·001,037	·001,035	·001,038
·207,303	·207,262	·207,346	·000,749	·000,748	·000,751
·185,709	·185,670	·185,745	·000,529	·000,528	·000,531
·165,953	·165,915	·165,987	·000,364	·000,363	·000,365
·147,915	·147,878	·147,944	·000,243	·000,243	·000,244
·131,480	·131,444	·131,507	·000,156	·000,156	·000,157
·116,539	·116,504	·116,562	·000,096	·000,096	·000,097
·102,988	·102,954	·103,007	·000,056	·000,057	·000,057
·090,727	·090,694	·090,745	·000,031	·000,031	·000,031
·079,663	·079,631	·079,675	·000,015	·000,016	·000,016
·069,706	·069,673	·069,716	·000,007	·000,007	·000,007
·060,769	·060,738	·060,778	·000,003	·000,003	·000,003
·052,773	·052,743	·052,778	·000,001	·000,001	·000,001
·045,641	·045,612	·045,646	·000,000	·000,000	·000,000
·039,302	·039,275	·039,304			

With the possible exception of the first example, in which the number of significant terms is small, the goodness of fit of the Pearson curves to the above series is quite close. The fit improves rapidly as the number of significant terms of the series increases, and when this number is fairly large, or, what is equivalent, when the standard deviation of the series is fairly large, the agreement between the terms of the series and the corresponding areas under the Pearson curve fitted to it by moments is sufficiently close to justify our replacing the Probability Integral of the series by that of the Pearson curve.

In the first example (p. 314), where the fit is not very close, the standard deviation is small, being approximately 2. In the following two examples the standard deviations of the series concerned lie between 3 and 3·5 and the corresponding fit has improved appreciably, giving an accuracy of three to four figures in the areas. This in itself is sufficiently close for most statistical purposes. In

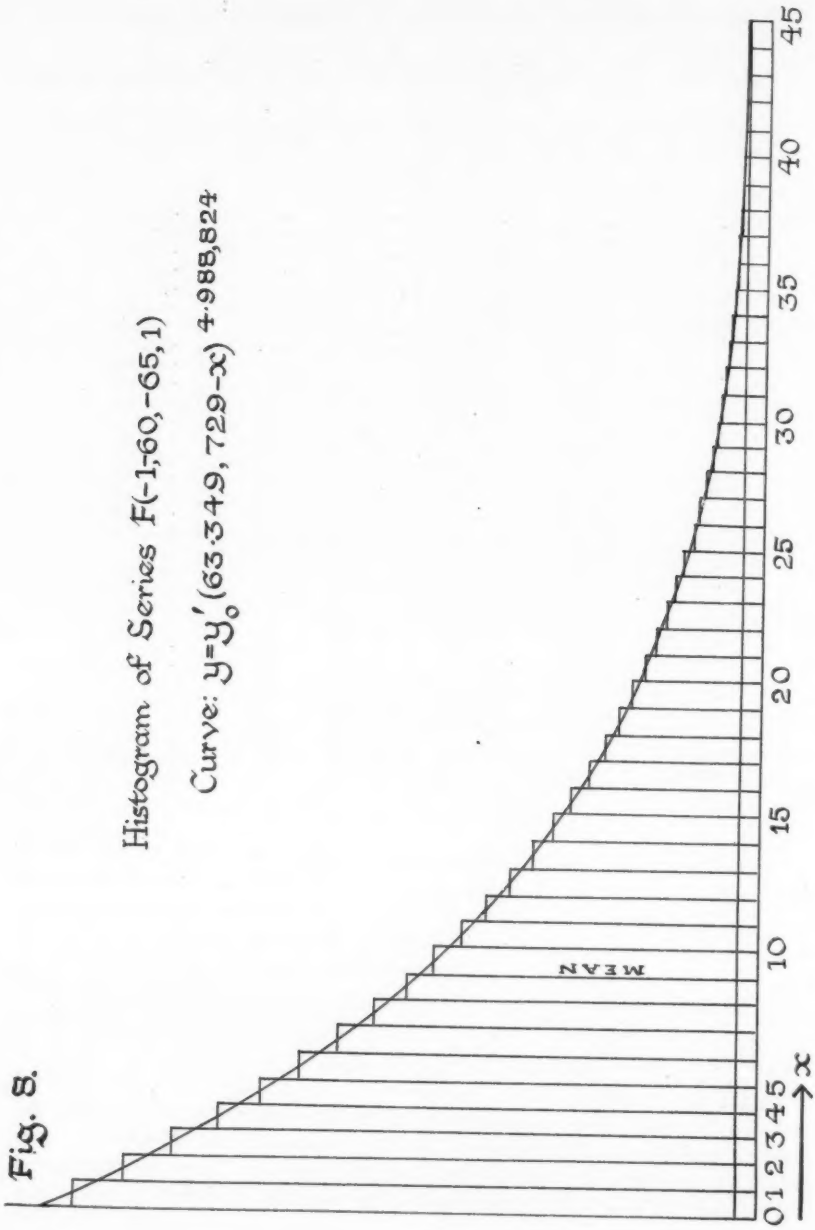

Histogram of Series F(-1,60,-65,1)
Curve: $y=y_0'(63{\cdot}349,729-x)^{4{\cdot}988,824}$
MEAN
0 1 2 3 4 5
10
15
20
25
30
35
40
45
x

Fig. 8.

the last example, the standard deviation is larger, being approximately 7·9. The resulting fit, with the exception of the extreme tail, is surprisingly close.

The number of significant terms of the hypergeometric series is approximately equal to six times the standard deviation. For series with less than 24 significant terms—corresponding to S.D.'s less than 4—one would not in actual practice go into the labour of fitting a Pearson curve in order to determine an approximation to the sum of a number of terms because this sum can be obtained accurately and quite readily by calculating the terms directly. It is doubtful, however, whether such a procedure would be adopted if the number of terms is greater than 30. The total sum of the series is, of course, known. Hence, the probable limit where one would calculate the terms directly is a series with 60 significant terms, corresponding to a standard deviation of about 10. For such series, the appropriate Pearson curve should give at least as good a fit as in the previous example (Ex. IV). For instance, the Pearson curve fitted by moments to the series $F(1, -60, -65, 1)$ is (p. 318)

$$y = y_0 x^{-\cdot 4001\cdot 138}(63\cdot 349{,}729 - x)^{4\cdot 988\cdot 8238} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}bis.$$

The ratio of the sum of the remainder of the series after the ninth term, to the total sum of the series, is ·399,3917.

The approximate value of this ratio found from the curve (i) is

$$\int_9^{63\cdot 349\cdot 729} y\,dx \Big/ \int_0^{63\cdot 349\cdot 729} y\,dx.$$

This is the incomplete beta function

$$I_{\cdot 857,9315}(5\cdot 988{,}8238;\ \cdot 999{,}8862).$$

By triple interpolation into the beta function tables, using third differences, this was found to be ·399,3800.

This differs from the true value by less than unity in the fifth place.

In order to appreciate what happens when the number of significant terms is large, the goodness of fit of a Pearson curve to the series $F(10{,}000;\ 10{,}000;\ 1;\ 1)$ is examined within a short range after the mode.

Sum of series $N = \cdot 224{,}5596 \times 10^{6019}$.

F is symmetrical and β_2 is approximately equal to 3.

$\mu_2 = 1249\cdot 979170$ whence $\sigma = 35\cdot 356{,}0445$.

In the following table the terms of the series after the mode are compared with the areas under the normal curve

$$y = \frac{N}{\sqrt{2\pi}\,.\,\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

which has the same mean standard deviation and sum as the above series.

The quantities tabulated are equal to 10^{-0017} times their actual values.

Terms of series	Areas under normal curve
·253,393	·253,382
·253,292	·253,281
·252,988	·252,977
·252,488	·252,471
·251,777	·251,765
·250,872	·250,861
·249,771	·249,760
·248,475	·248,465
·246,989	·246,978
·245,316	·245,304
·243,458	·243,446
·241,422	·241,410
·239,211	·239,200
·236,831	·236,820

The true value of β_2 is 2·999906, which is slightly less than 3. This accounts for the constant deficiency of 1 in the fifth place in the areas under the normal curve. A slightly better fit is given by the Type II curve which has the same first four moments as the series. Even so, the normal curve gives sufficiently accurate results.

I am indebted to Professor Pearson for suggesting this problem to me and for his advice and criticism throughout the preparation of this paper.

THE BODY BUILD OF AMERICAN-BORN JAPANESE CHILDREN.

By P. M. SUSKI, M.D.

The better bodily development of American-born Japanese children over the children born in Japan, presumably through environmental influences, is the fact observed by many during the past few years. Ishiwara first called attention, in Japanese medical literature, to the superior height and weight of the second generation Japanese in America. A few years ago, I measured American-born young Japanese ranging in ages from 15 to 25. At the request of Dr Ishiwara, two years ago, I measured a hundred Japanese boys born in America. In the former group the height and weight only were obtained. In the latter, the stature, sitting height, iliac spine height, knee joint height and weight were measured. The ages of these boys were 19 to 22. With the measurements of the latter group, the height and weight of their parents were compared, and it has been found that the children born in America excelled their parents by about 5% in height and about 6% in weight. (In several instances, one or both of the parents could not be measured.)

That there is a marked difference in bodily development of the children born and raised under environments foreign to those under which the parents thrived, is found to be true by many authors. American-born descendants of immigrants differ in type from the foreign-born parents, according to Boas(1) who further asserts that, the longer the sojourn of parents in America before the children are born, the more intense the influence is felt. Bakwin and Bakwin(2) demonstrated a striking difference in the body build of infants of two groups from different social environments. Gray and Gower(3) found the height and weight of girls in private schools are better than the average. Studying the bodily growth of Chinese children in Hawaii and those in Chekiang and Kiangsu Provinces in China, Appleton(4) observes a marked influence in the manner of growth and development, such that the growth curves, in general, are more smooth and regular, the period of growth retardation comes in a later period and to a lesser degree in Hawaiian-born Chinese children than those born in China. She thinks that the acceleration of growth in the stature of Chinese boys in Hawaii which is greater than that of boys in East China is mainly due to the more rapid increase in length of lower limbs, in which the height of knee is the more active factor.

If lands foreign to the parents act upon the growth and development of their offspring, we may enquire what may be the influence of America (i.e. Southern California) upon the growth of the children of Japanese parents; this was the

question I had always borne in mind while making measurements of children in Los Angeles, born of Japanese parents.

The material consists of students of private Japanese schools, which are conducted to supplement the public school education of the city of Los Angeles. So, practically, all of the children are public school children. The age range is from 6 to 19, although the youngest and those above 18 years of age are so few in number, that I have omitted them from the tables and charts. The parents of the children are from all walks of life, from unskilled labourers to business men, manufacturers or literary and professional people. With the exception of about one per cent. of the children, who were born in Hawaii, those measured are California born.

These are the first of the series of the intended annual physical measurements of Japanese children born in America. As yet the number is not large enough to show conclusive results, but the chief reasons for the present publication of my findings are to show at least the apparent marked difference in stature, weight, chest circumference, sitting height and leg length which I found, when compared with figures available for the Japanese children in Japan, and also to request the authorities in this line of work to point out any defects in procedure in my work, and to give me suggestions for research work in directions which I may not have taken.

The children will be measured hereafter once a year at the same time of the year as far as possible. Thus far, the measurements were limited to the months of June and July. But the field will be enlarged so that other lots of children will be obtained for other months of the year. That the growth rate is not uniform throughout the year, is pointed out by Orr and Clark (5), Zeiner-Hendriksen (6), Nylin (7) and others. Hejinian and Hatt (8) made monthly measurements of stem-stature index. Sumner and Whitacre (9) followed monthly change in weight of Texas school children. Iowa Child Welfare Research Station (10) reports various measurements made monthly of young children from 3 to 6 years of age. It is usual for the investigators to take measurements weekly or monthly in the case of infants or small children. The figures obtained through these frequent measurements, if computed individually, would certainly reveal the seasonal variation in growth and development of children. But the purpose of my present investigation concerns the annual growth rate. I have planned the examination of all children only once a year, and at the same season of the year for each group of children, so that the seasonal variation in growth would not interfere with results.

The time of measurement was in the afternoon for one school and in the forenoon for others. Martin (11) recommends about 10 A.M. as the ideal time of measurement. Nakadate found the maximum diurnal variation in stature of children of 10 to 15 years of age to be 1·66 cm. for boys, and 1·52 cm. for girls. He finds the maximum in stature to be immediately after rising in the morning, and it drops to the lower level around 11.30 A.M.; from then on till 8 P.M. there is a very small amount of change (12).

The nearest birthday system is employed in classifying the children into different age groups; i.e., the children from 6 years 6 months to 7 years 6 months of age are grouped as 7 years, and so on. When making comparison with figures obtained from Japan, it was necessary to make allowance of a half year, because of the fact that many Japanese investigators classify a child between the 6th and 7th birthday as 7 years old, and so on.

Instruments used are: Martin's anthropometric set, consisting of a measuring rod with arms, callipers, steel tape measure and a slide compass; a box with a square flat top to measure sitting height; a standard spring scale.

I have examined the lists of measurements of Martin(11), Hrdlicka(13), the Geneva Agreement as cited by Hrdlicka(13), Yoshida(12), and Lucas and Pryor(23), and found them to contain 30 to 70 direct measurements of the living body, exclusive of functional and capacity tests. As Hrdlicka points out, the number of practicable measurements on the human body is infinite, and these measurements may be of value if taken by the same method on a sufficiently large number of individuals of various groups. Bearing these things in mind, I made a selection of physical measurements, chiefly through the suggestions of Ishiwara, namely, ten measurements, nine direct and one indirect. They were: 1. Stature, 2. Sitting height, 3. Height of anterior superior spine of ilium, 4. Leg length, 5. Knee joint height, 6. Arm span, 7. Chest circumference, 8. Intercristal width, 9. Acromial width, 10. Body weight.

The stature is measured with a child standing on a flat floor without shoes, and the head erect, keeping the eye-ear line horizontal. The left anterior superior spine of ilium is first determined by palpation, marked with a dermatograph pencil and then measured. Downes(14) reports there are over 50 % of asymmetry in the height of the iliac spine exceeding 0·5 cm. I found only a few cases of asymmetry, and these in a slight degree.

For the knee joint height, a groove between femur and tibia is easily found laterally on the level between the patellar prominence and the tibial tuberosity. By inspection from the left side of the left knee joint, one may easily locate the position of the groove, and verify it by palpation and mark it with a pencil. The distance from the floor is then determined. Martin recommends the internal groove of the knee joint, and it is most subcutaneous there and therefore easiest to measure. But I did not meet any difficulty in locating the groove on the lateral side.

The leg length has been computed from the figures of the anterior superior spine height according to Martin's formula, and is intended to measure the height of the head of femur from the floor, corresponding to Martin's "ganze Beinlänge" (the entire leg length).

The sitting height is obtained by seating a child on a box not too high so that the child can rest the feet on the floor, and measuring the height of the vertex above the seat level. Gray and Root(15) call our attention to the fact that some workers call this and other measurements under various names as stem-length, trunk-length,

rump-length, etc., with analogous terms in other languages, so that it may mean lengths from vertex, shoulder, suprasternal notch, 7th cervical vertebra or acromion down to os pubis, first or last coccyx, perineum, ischial tuberosities or gluteal lines. It is therefore necessary for one to be very careful when referring to any old literature, and it is also very important to make a clear statement as to exactly how the sitting height is obtained. The method I employed is in accord with that of the Geneva Agreement of 1912, used by Hrdlicka, Martin, v. Pirquet, Yoshida, Ishiwara and others. Dreyer(16) urges bending of the knee and pulling up slightly, instead of square sitting with both knee and hip joints at right angles. I tried Dreyer's method in a few of the children after measuring them in the usual way, but I failed to find the 3% difference as pointed out by Dreyer. In the first place, it was difficult to keep the sacrum in contact with the post, if the knees were drawn up as in Dreyer's illustration, besides in nearly all of my cases (which were young children) the ischial tuberosities were quite subcutaneous with no appreciable amount of adipose tissue intervening.

The arm span is measured with a steel tape measure, from the tip of one middle finger to the other, the arms being stretched to both sides of the body horizontally. I found it was the best way to let a child stand against the wall to take this measurement.

The chest circumference is taken at the middle of the normal inspiration and expiration, on the level above nipples in front and below the scapular angle on the back. The bicristal width is obtained with a pair of callipers, the tips of which were firmly pressed on the widest part of the pelvic crest The acromial width is measured in a similar manner, only here as the respiration, shrugging of shoulders or throwing backward or forward of the entire shoulder will give a great change in results. I exercised great care in taking the diameter at the middle of respiration and keeping the shoulders at the proper position.

The body weight is measured nude, allowing only a garment weighing 200 gm., and after emptying the bladder. It was recorded in the nearest $\frac{1}{4}$ pound, and later converted into kilograms.

The arithmetic mean, for each age sex group, of all measurements is given in the Table I, with the standard deviation and probable error of each, computed according to the formulae:(17)

$$\sigma = \sqrt{\frac{\Sigma(fd^2)}{N} - \left(\frac{\Sigma(fd)}{N}\right)^2},$$

$$\text{P.E.} = \pm 0{\cdot}6745\,\frac{\sigma}{\sqrt{N}}.$$

Table II gives the relative measures to the total height or stature, often called the index to stature, or ratio. The relative measures are computed as follows:

$$\text{Index} = \frac{\text{measure other than stature} \times 100}{\text{stature}}.$$

TABLE I.

Measurements of American-born Japanese Children.

Boys (cm.)		Stature (cm.)		Sitting Height (cm.)		Ant. Super. Spine Height (cm.)		Leg Length (cm.)		Knee Joint Height (cm.)		Arm Span (cm.)		Chest Circumference (cm.)		Bicristal Width (cm.)		Acromial Width (cm.)		Weight (kg.)	
n	Age	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
31	7	115·5 ± ·49	4·1	64·3 ± ·23	1·8	61·9 ± ·29	2·4	60·4 ± ·29	2·4	32·5 ± ·18	1·5	114·9 ± ·49	4·1	57·9 ± ·28	2·3	18·1 ± ·08	0·6	24·5 ± ·12	1·0	22·3 ± ·15	1·2
61	8	121·3 ± ·46	5·2	66·1 ± ·22	2·6	66·3 ± ·36	4·1	64·8 ± ·20	2·3	30·7 ± ·20	2·3	120·3 ± ·50	5·8	55·8 ± ·27	3·1	18·4 ± ·00	0·0	25·4 ± ·11	1·2	23·9 ± ·21	2·4
76	9	126·2 ± ·46	5·2	68·2 ± ·20	2·6	69·3 ± ·29	3·7	67·7 ± ·29	3·8	32·3 ± ·15	2·0	125·5 ± ·44	5·7	61·0 ± ·28	3·6	19·2 ± ·07	0·9	26·2 ± ·12	1·6	26·5 ± ·29	3·8
65	10	131·6 ± ·43	5·4	70·6 ± ·25	3·0	73·7 ± ·37	4·4	71·5 ± ·34	4·1	34·6 ± ·18	2·1	131·7 ± ·53	5·3	63·7 ± ·32	3·8	19·9 ± ·11	1·3	27·5 ± ·19	2·3	29·7 ± ·36	4·3
52	11	135·9 ± ·65	6·9	72·3 ± ·34	3·6	76·3 ± ·43	4·6	74·3 ± ·42	4·5	35·5 ± ·25	2·7	136·1 ± ·78	8·3	66·6 ± ·51	5·4	20·4 ± ·15	1·6	27·9 ± ·18	1·9	32·8 ± ·45	4·8
51	12	139·7 ± ·68	7·2	74·2 ± ·33	3·5	78·7 ± ·47	5·0	76·7 ± ·44	4·7	36·7 ± ·24	2·5	141·1 ± ·68	7·2	70·5 ± ·46	4·8	20·9 ± ·14	1·4	29·2 ± ·19	2·0	34·8 ± ·52	5·4
54	13	147·3 ± ·70	7·7	77·7 ± ·35	3·9	82·9 ± ·44	4·8	80·5 ± ·41	4·5	39·7 ± ·27	2·9	149·3 ± ·78	8·5	71·9 ± ·43	4·7	22 0 ± ·16	1·8	30·5 ± ·14	1·5	39·5 ± ·54	5·8
37	14	155·6 ± ·79	7·2	81·9 ± ·46	4·2	87·8 ± ·46	4·1	85·0 ± ·45	4·0	41·6 ± ·28	2·6	158·7 ± ·99	8·9	77·1 ± ·56	5·1	23·9 ± ·23	2·1	32·5 ± ·26	2·4	46·3 ± ·78	7·0
32	15	161·3 ± ·92	7·7	83·9 ± ·45	3·8	90·4 ± ·47	3·9	87·2 ± ·44	3·7	42·8 ± ·29	2·4	163·6 ± ·90	7·2	79·7 ± ·48	4·0	24·3 ± ·22	1·8	33·3 ± ·24	2·0	50·0 ± ·69	5·8
26	16	161·4 ± ·59	4·5	85·6 ± ·34	2·5	90·7 ± ·51	3·9	87·6 ± ·51	3·8	42·9 ± ·33	2·5	164·1 ± ·74	5·6	80·7 ± ·43	3·3	24·6 ± ·20	1·5	34·5 ± ·26	2·0	53·0 ± ·78	5·9
13	17	161·4 ± 1·23	4·6	86·6 ± ·48	2·6	89·9 ± ·72	3·8	86·8 ± ·73	3·9	42·4 ± ·39	2·1	165·7 ± 1·25	6·7	85·1 ± 1·28	6·8	24·9 ± ·30	1·6	34·7 ± ·29	1·6	56·8 ± 1·46	7·8
Girls																					
56	7	115·6 ± ·40	4·5	63·9 ± ·24	2·7	62·4 ± ·27	3·0	60·9 ± ·27	3·0	29·5 ± ·14	1·5	113·0 ± ·53	5·8	57·2 ± ·27	3·0	17·9 ± ·00	0·0	24·8 ± ·11	1·2	21·1 ± ·25	2·8
89	8	119·4 ± ·47	5·3	65·6 ± ·21	2·9	65·2 ± ·28	3·9	63·7 ± ·26	3·6	30·7 ± ·30	2·1	118·4 ± ·46	6·5	58·8 ± ·22	3·1	18·9 ± ·08	1·2	25·5 ± ·12	1·6	22·6 ± ·23	3·3
82	9	124·1 ± ·40	5·4	67·3 ± ·24	3·2	68·0 ± ·31	4·1	66·3 ± ·30	4·1	31·9 ± ·15	2·0	122·1 ± ·45	6·0	60·4 ± ·22	3·0	19·4 ± ·08	1·0	26·4 ± ·13	1·8	24·6 ± ·25	3·3
74	10	130·4 ± ·44	3·6	69·8 ± ·24	3·0	72·9 ± ·28	3·6	71·1 ± ·27	3·4	33·9 ± ·15	1·9	130·9 ± ·58	7·4	62·5 ± ·29	3·7	20·1 ± ·11	1·4	27·4 ± ·13	1·7	28·2 ± ·31	4·0
61	11	135·4 ± ·55	6·3	72·4 ± ·31	3·6	76·1 ± ·36	4·2	74·2 ± ·36	4·1	35·5 ± ·21	2·4	136·0 ± ·61	7·1	64·8 ± ·44	5·0	20·9 ± ·13	1·5	28·5 ± ·13	1·5	30·2 ± ·42	4·9
54	12	143·2 ± ·54	5·8	76·3 ± ·37	4·0	80·4 ± ·30	3·3	78·3 ± ·29	3·1	37·7 ± ·19	2·1	144·7 ± ·61	6·7	70·3 ± ·59	6·4	22·6 ± ·14	1·5	30·6 ± ·17	1·9	38·1 ± ·82	8·9
58	13	146·0 ± ·44	4·9	78·3 ± ·29	3·3	81·3 ± ·31	3·6	79·2 ± ·30	3·4	37·9 ± ·18	2·0	147·5 ± ·53	6·0	72·4 ± ·42	4·7	23·3 ± ·13	1·5	31·3 ± ·15	1·7	39·8 ± ·50	5·7
49	14	149·4 ± ·41	4·2	81·1 ± ·28	2·9	83·0 ± ·31	3·2	80·7 ± ·30	3·1	38·8 ± ·23	2·4	150·9 ± ·44	4·6	74·2 ± ·45	4·7	23·9 ± ·16	1·6	32·5 ± ·15	1·5	43·1 ± ·52	5·4
34	15	150·4 ± ·55	4·7	81·5 ± ·41	3·5	83·8 ± ·37	3·2	81·3 ± ·35	3·0	38·9 ± ·22	1·9	153·3 ± ·60	5·1	74·9 ± ·57	4·9	24·6 ± ·17	1·5	32·6 ± ·13	1·2	43·4 ± ·84	7·3
16	16	151·8 ± ·78	4·8	83·3 ± ·39	2·4	83·8 ± ·71	4·4	81·3 ± ·67	4·1	39·0 ± ·36	2·2	153·9 ± 1·00	6·1	74·8 ± ·54	3·3	24·8 ± ·14	0·9	32·5 ± ·22	1·3	45·7 ± ·73	4·4

TABLE II.

American-born Japanese Children. Measurements relative to Stature.

Age Boys	Stature	Sitting Height	Ant. Sup. Spine	Leg Length	Knee Joint	Arm Span	Chest Circ.	Crist. Width	Acrom. Width
7	100	55·7	53·6	52·3	25·3	98·7	50·1	15·67	21·21
8	100	54·5	54·7	53·4	25·3	99·2	48·5	15·17	20·94
9	100	54·0	54·9	53·7	25·6	99·4	49·0	15·21	20·76
10	100	53·6	56·0	54·6	26·3	100·0	48·4	15·12	20·90
11	100	53·2	56·1	54·7	26·1	100·1	49·0	15·01	20·53
12	100	53·1	56·3	54·6	26·3	101·0	50·5	14·96	20·90
13	100	52·7	56·3	54·9	26·9	101·6	48·8	14·94	20·71
14	100	52·6	56·4	54·5	26·7	102·0	49·6	15·36	20·89
15	100	52·0	56·0	54·2	26·5	101·4	49·4	15·07	20·65
16	100	53·0	56·2	54·3	26·6	101·6	50·0	15·24	21·38
17	100	53·6	55·6	53·7	26·2	102·5	52·7	15·41	21·47
GIRLS									
7	100	55·3	54·0*	52·7*	25·5*	97·8	49·5	15·48	21·45*
8	100	54·9*	54·6	53·4	25·7*	99·2	49·2*	15·83*	21·36*
9	100	54·2*	54·8	53·6	25·7*	98·4	48·7	15·63*	21·27*
10	100	53·5	55·9	54·8*	26·0	100·4*	47·9	15·41*	21·01*
11	100	53·5*	56·2*	55·1*	26·2*	100·4*	47·9	15·44*	21·05*
12	100	53·3*	56·1	54·7*	26·3	101·0	49·1	15·78*	21·37*
13	100	53·6*	55·7	54·3	26·0	101·0	49·6*	15·96*	21·44*
14	100	54·3*	55·6	54·2	26·0	101·0	49·7*	16·00*	21·75*
15	100	54·2*	55·7	54·4*	25·9	101·9*	49·8*	16·36*	21·68*
16	100	54·9*	55·2	53·2	25·7	101·4	49·3	16·34*	21·41*

* The asterisk shows that in these cases the girls' figures were higher than the boys.

Tables III and IV together with Figures 1 to 10 show the American, German, Japanese, and American-born Japanese figures. The American figures

TABLE III.

Standing Height by different Authorities compared.

BOYS Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.	GIRLS Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.
7	116·7	115·0	115·5	7	116·1	113·0	115·6
8	121·9	120·0	121·3	8	121·4	118·0	119·4
9	126·8	125·0	126·2	9	125·8	123·0	124·1
10	132·0	130·0	131·6	10	130·9	128·0	130·4
11	136·3	135·0	135·9	11	136·1	133·0	135·4
12	141·0	140·0	139·7	12	143·5*	139·0	143·2*
13	146·6	145·0	147·3	13	148·4*	146·0*	146·0
14	153·6	151·0	155·6	14	152·7	153·0*	149·4
15	159·2	157·0	161·3	15	155·6	158·0*	150·4
16	165·9	—	161·4	16	156·5	160·0	151·8

* Those marked with an asterisk are instances in which girls excel boys in height.

are obtained by combining the United States Government Table of Heights and Weights and Holt's Age-Height-Weight Table (18). The German figures are from the

TABLE IV.

Body Weight by different Authors compared.

BOYS Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.	GIRLS Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.
7	22·3	23·0	22·3	7	21·7	21·0	21·1
8	24·6	25·0	23·9	8	23·9	23·0	22·6
9	27·1	27·5	26·5	9	26·0	25·0	24·6
10	29·9	30·0	29·7	10	28·7	27·0	28·2
11	32·4	32·5	32·8	11	31·6	29·0	30·2
12	35·4	35·0	34·8	12	36·2*	32·0	38·1*
13	39·3	37·5	39·5	13	40·8*	37·0	39·8*
14	44·0	41·0	46·3	14	45·1*	43·0*	43·1
15	49·4	45·0	49·9	15	48·6	48·0*	43·4
16	55·5	—	53·0	16	51·1	52·0	45·7

* Those marked with an asterisk are instances in which girls excel boys in weight.

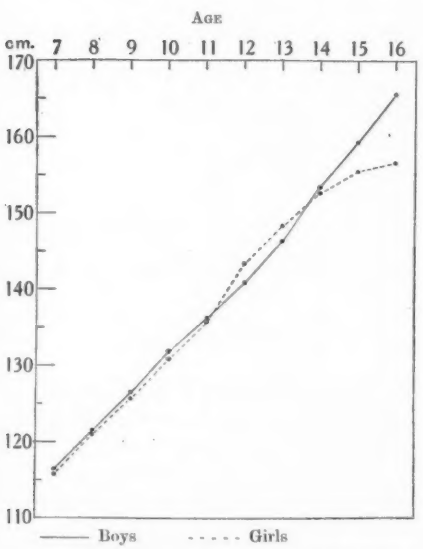


Fig. 1. Standing Height of Americans. (U.S. Government figures and Holt's combined.)

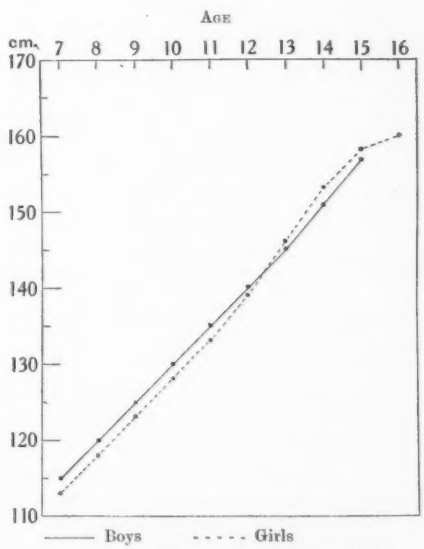


Fig. 2. Standing Height of European Children. (Camerer-v. Pirquet figures.)

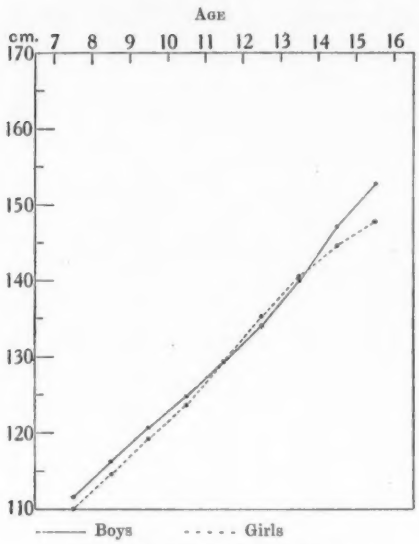


Fig. 3. Standing Height of Japanese. (Education Ministry figures.)

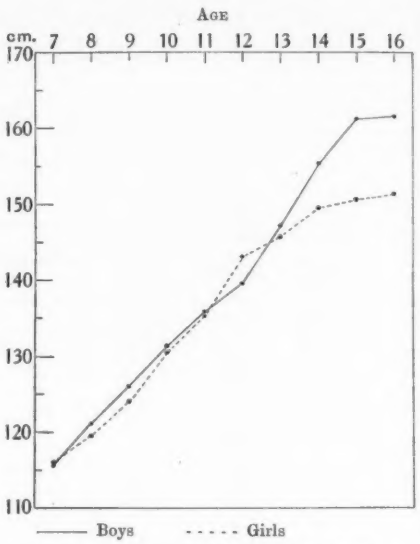


Fig. 4. Standing Height of American-born Japanese.

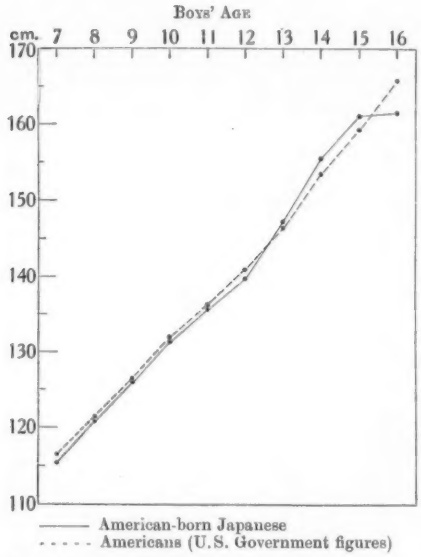


Fig. 5. Standing Height of American-born Japanese and Americans compared.

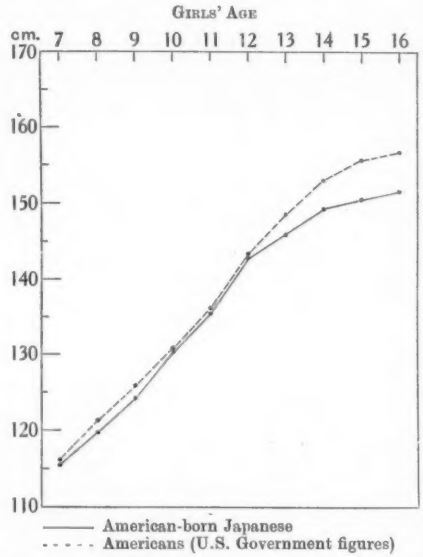


Fig. 6. Standing Height of American-born Japanese and Americans compared.

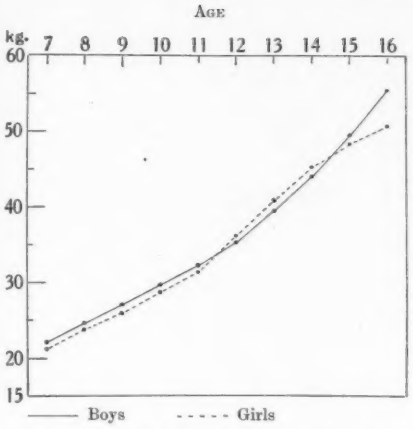


Fig. 7. Body Weight of Americans. (Holt's and U.S. Government figures combined.)

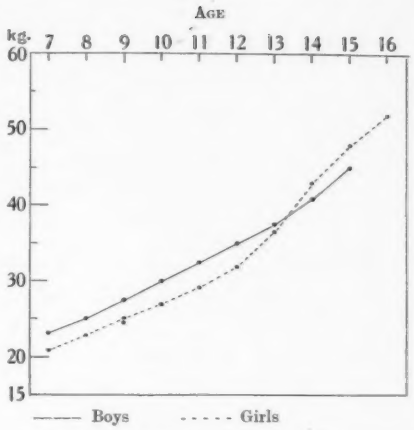


Fig. 8. Body Weight of European Children. (Camerer-v. Pirquet figures.)

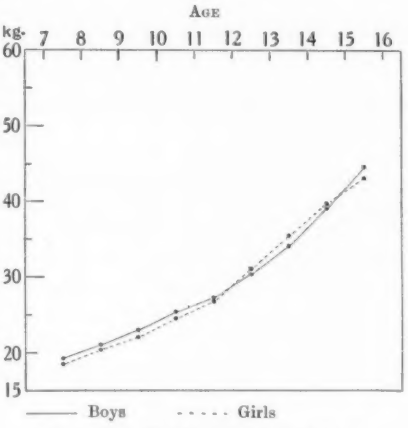


Fig. 9. Body Weight of Japanese Children. (Education Ministry figures.)

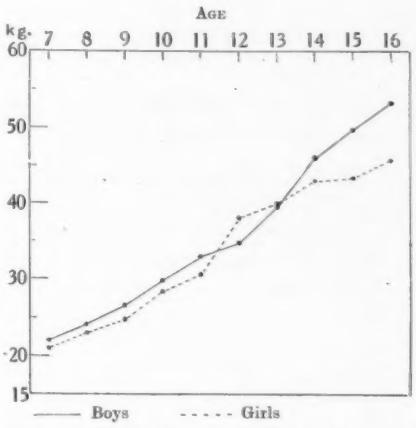


Fig. 10. Body Weight of American-born Japanese.

Age-Length-Weight Table by Camerer and v. Pirquet (19). The Japanese figures are from the table issued by the Japanese Ministry of Education (20). As seen in Figs. 5 and 6, the stature of American-born Japanese boys is very close to that of American boys at least up to 16 years of age while there exists a wide difference between the former and that of Japan-born Japanese (see Fig. 11). The stature of American-born Japanese girls is nearly the same as that of American girls up to the age of 12, after which it drops off a little, but when compared with the figures of Japan-born Japanese girls, there is quite a space above the latter. The American-born

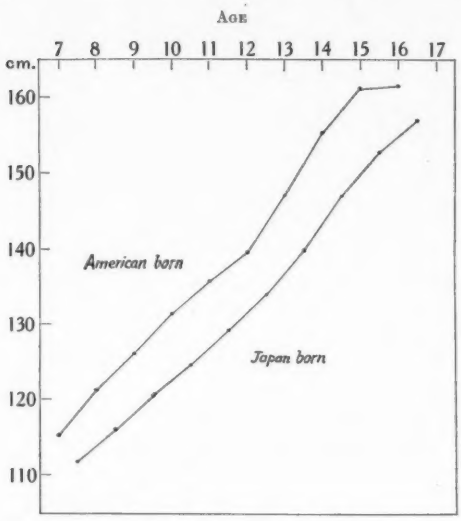


Fig. 11. Standing height of American-born Japanese Boys compared with Japan-born Japanese Boys.

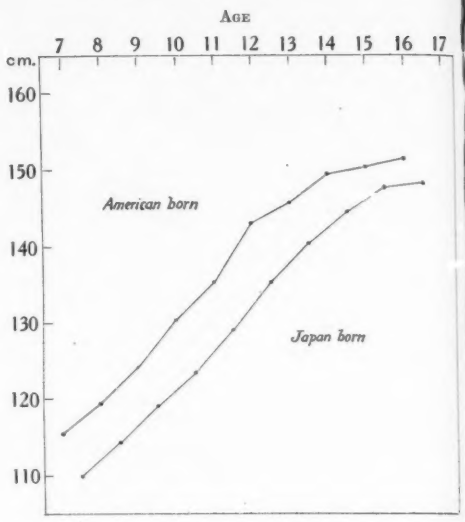


Fig. 12. Standing Height of American-born Japanese Girls compared with Japan-born Japanese Girls.

Japanese also excel their cousins in Japan in body weight and chest circumference (see Table V and Figs. 13—16).

It is a well-known fact that the children from infancy to 11 and 12 years of age, grow up without showing a marked difference with respect to sex. Between 11 and 14, corresponding to the pre-puberty period, girls excel boys in stature and weight, after which boys attain superior weight and height. Our tables and figures show these changes in all groups. But the relationship between girls' curves and boys' curves, which is of slightly different character in different groups, is best studied on the special diagram I have prepared for the purpose. First, I took the boys' height or weight at each age as a standard and figured out a percentage for girls at that age. Thus, for the stature,

$$\frac{\text{Girls' Mean Stature at given age} \times 100}{\text{Boys' Mean Stature at the same age}},$$

and called it "the percentage of girls' stature on that of boys." In the same manner, the body weights of boys and girls are compared (Tables VI and VII, Figs. 17 and 18). The boys' figures are represented by a straight line at the 100% level, and the girls' curve runs below it while the boys' figures are greater, but when the girls' figures surpass that of boys, the curve runs above the boys' line. When I first drew this type of diagram, I showed it to Drs Lucas and Pryor, who thought it was unique and excellent in indicating the change in relationship of boys' and girls' figures so clearly. Later, however, I found Collins and Clark (21) had already devised

TABLE V.

Comparison between Japan-born and American-born Japanese Children in Stature, Chest Circumference and Body Weight.

Age	Height (cm.)		Chest Circumference (cm.)		Weight (kg.)	
BOYS	Japan born	U.S. born	Japan born	U.S. born	Japan born	U.S. born
7	—	115·5	—	57·9	—	22·3
7½	111·7	—	56·1	—	19·3	—
8	—	121·3	—	58·8	—	23·9
8½	116·1	—	57·9	—	21·1	—
9	—	126·2	—	61·9	—	26·5
9½	120·6	—	60·0	—	23·1	—
10	—	131·6	—	63·7	—	29·7
10½	124·8	—	61·8	—	25·1	—
11	—	135·9	—	66·6	—	32·8
11½	129·1	—	63·9	—	27·4	—
12	—	139·7	—	70·5	—	34·8
12½	133·9	—	65·5	—	30·1	—
13	—	147·3	—	71·9	—	39·5
13½	140·0	—	68·4	—	34·0	—
14	—	155·6	—	77·1	—	46·3
14½	147·3	—	71·6	—	39·3	—
15	—	161·3	—	79·7	—	49·9
15½	152·7	—	75·1	—	44·8	—
16	—	161·4	—	80·7	—	53·0
16½	157·3	—	77·9	—	48·6	—
17	—	161·6	—	85·1	—	56·8
17½	159·4	—	79·7	—	51·2	—
GIRLS						
7	—	115·6	—	57·2	—	21·1
7½	110·0	—	54·2	—	18·5	—
8	—	119·4	—	58·8	—	22·6
8½	114·5	—	55·8	—	20·3	—
9	—	124·1	—	60·4	—	24·6
9½	119·1	—	57·6	—	22·2	—
10	—	130·4	—	62·5	—	28·2
10½	123·6	—	59·4	—	24·4	—
11	—	135·4	—	64·8	—	30·2
11½	129·1	—	61·8	—	27·3	—
12	—	143·2	—	70·8	—	38·1
12½	135·1	—	64·5	—	31·0	—
13	—	146·0	—	72·4	—	39·8
13½	140·4	—	67·6	—	35·2	—
14	—	149·4	—	74·2	—	43·1
14½	144·8	—	71·2	—	39·6	—
15	—	150·4	—	74·9	—	43·4
15½	147·6	—	73·9	—	43·1	—
16	—	151·8	—	74·8	—	45·7
16½	148·2	—	75·8	—	45·4	—

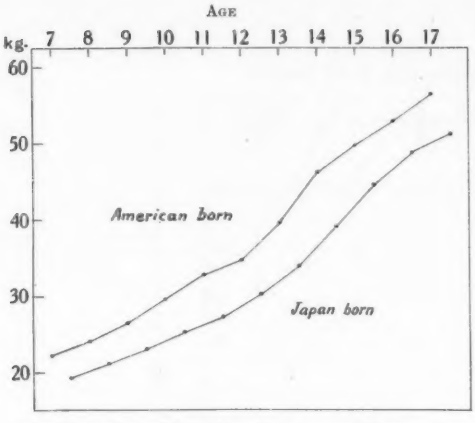


Fig. 13. Body Weight of American-born Japanese Boys compared with Japan-born Japanese Boys.

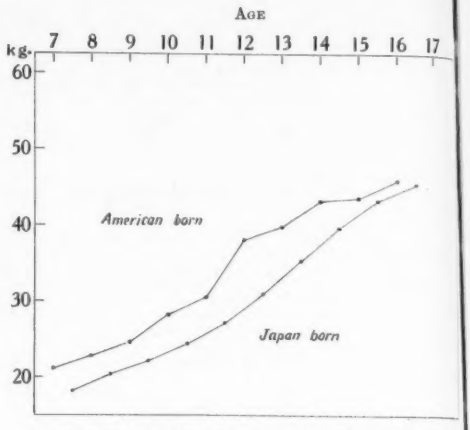


Fig. 14. Body Weight of American-born Japanese Girls compared with Japan-born Japanese Girls.

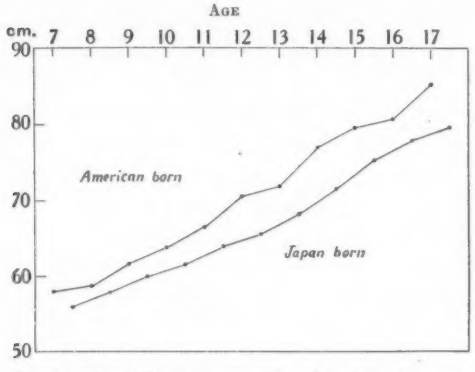


Fig. 15. Chest Circumference of American-born Japanese Boys compared with Japan-born Japanese Boys.

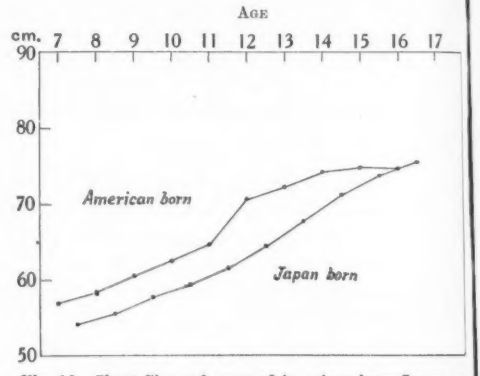


Fig. 16. Chest Circumference of American-born Japanese Girls compared with Japan-born Japanese Girls.

a somewhat similar diagram in 1929*. I have also figured out the percentage of girls' other relative values on those of boys, namely, the relative sitting height, relative iliac spine height, relative leg length, relative knee joint height, relative arm span, relative chest circumference, relative bicristal width, relative acromial width. Among these, I could only make comparison with values of Japanese in relative sitting height, relative leg length, and relative chest circumference, apart from stature and weight (see Tables VI, VII and VIII, also Figs. 17, 18 and Figs. 19 *b*—26 *b*).

* [This seems merely a diagram of the customary sex-ratio multiplied by 100; the sex-ratio is very familiar to anthropometricians and craniometricians. Ed.]

TABLE VI.

The Percentage of Girls' Standing Height to that of Boys.

Age	U.S. Gov. and Holt's Figures combined Americans	Camerer-v. Pirquet Germans	American-born Japanese	Education Ministry Japanese
7	99·5	98·3	100·1	98·5
8	99·6	98·3	98·4	98·6
9	99·2	98·4	98·3	98·8
10	99·2	98·5	99·1	99·0
11	99·9	98·5	99·6	100·0
12	101·9	99·3	102·5	100·9
13	101·4	100·3	99·1	100·2
14	99·4	101·3	96·0	98·3
15	97·7	100·6	93·2	96·7
16	94·4	—	94·0	94·2

TABLE VII.

The Percentage of Girls' Weight to that of Boys.

Age	U.S. Gov. and Holt's Figures combined Americans	Camerer-v. Pirquet Germans	American-born Japanese	Education Ministry Japanese
7	97·1	91·3	94·6	95·9
8	98·3	92·0	94·6	96·2
9	96·2	90·9	92·8	96·1
10	96·0	90·0	95·8	97·2
11	97·4	89·2	92·1	99·6
12	102·3	91·4	109·5	103·0
13	104·0	98·7	100·8	103·5
14	102·4	104·9	93·1	100·8
15	98·4	106·7	87·0	96·2
16	91·9	—	86·0	93·4

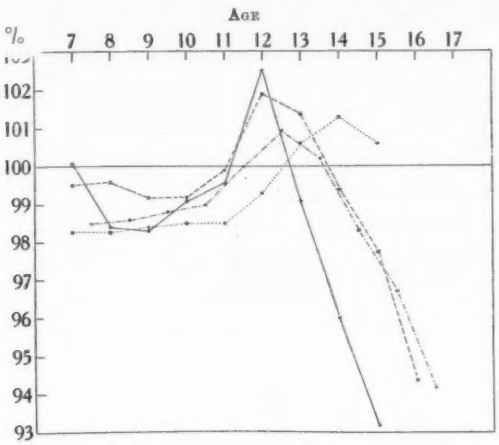


Fig. 17. The Percentage of Girls' Standing Height to that of Boys.

This diagram shows percentage values of girls' standing height, taking the boys' standing height as the standard or 100 % for each age. The figures published by the Ministry of Education of Japan are based upon age groups taken from one birthday to the next. For this reason, the points are marked on the diagram half-way between ages.

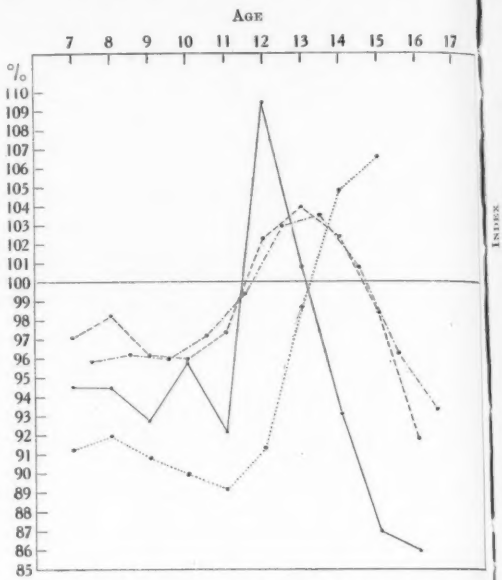


Fig. 18. The Percentage of Girls' Body Weight to that of Boys.

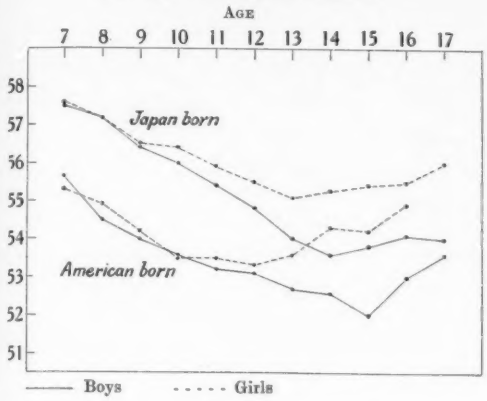
German (Camerer-v. Pirquet)
American (U.S. Government-Holt's)
Japanese (Education Ministry)
Japanese (American-born)

TABLE VIII. *Comparison of Relative Sitting Heights of Japan-born and American-born Japanese Children.*

Age	Japanese in Japan (Figures of Drs Tsurumi, Nakadate, Yagi and Toyoda combined)		American-born Japanese	
	Boys	Girls	Boys	Girls
7	57·5	57·6*	55·7	55·3
8	57·2	57·2	54·5	54·9*
9	56·4	56·5*	54·0	54·2*
10	56·0	56·4*	53·6	53·5
11	55·4	55·9*	53·2	53·5*
12	54·8	55·5*	53·1	53·3*
13	54·0	55·1*	52·7	53·6*
14	53·6	55·3*	52·6	54·3*
15	53·8	55·4*	52·0	54·2*
16	54·1	55·5*	53·0	54·9*
17	54·0	56·0*	53·6	—

* Those marked with an asterisk are instances in which girls excel boys in sitting height.

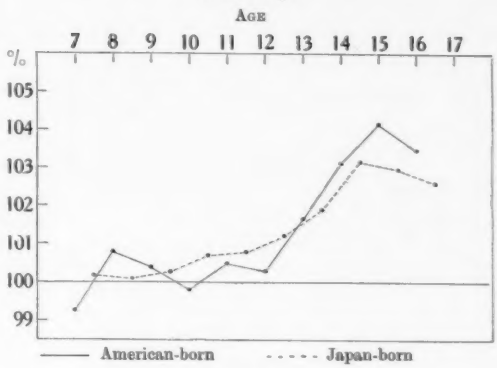
Relative Sitting Height of American-born Japanese and Japan-born Japanese compared



Boys ----- Girls

Fig. 19 *a*.

Percentage of Girls' Relative Sitting Height to that of Boys

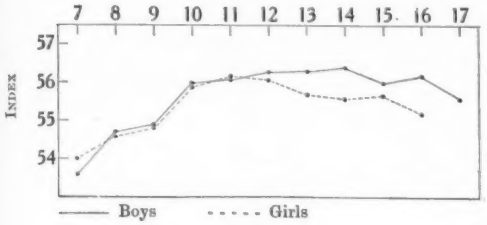


American-born ----- Japan-born

Fig. 19 *b*.

Relative Height of Anterior Superior Iliac Spine

$$\frac{\text{Iliac Spine Height}}{\text{Stature}} \times 100$$



Boys ----- Girls

Fig. 20 *a*.

Percentage of Girls' Relative Height of Anterior Superior Iliac Spine to that of Boys

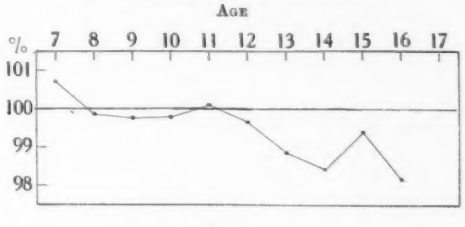
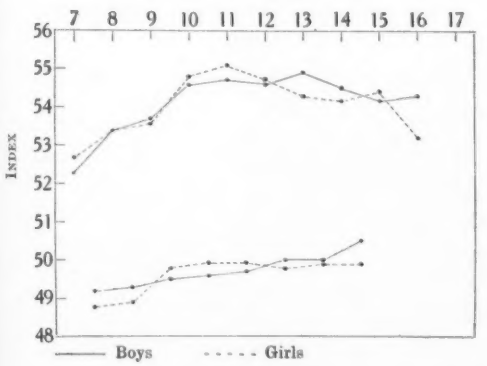


Fig. 20 *b*.

American-born Japanese.

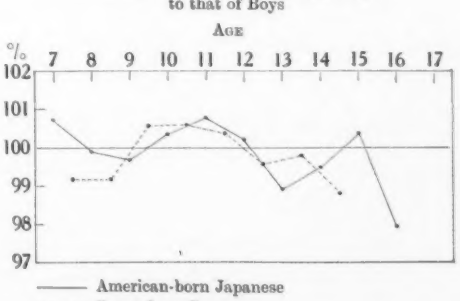
Relative Leg Length, or $\frac{\text{Leg Length}}{\text{Stature}} \times 100$



Boys ----- Girls

Fig. 21 *a*.

Percentage of Girls' Relative Leg Length to that of Boys



American-born Japanese

Japan-born Japanese

(These figures are measurements of Drs Mishima and Minayoshi combined.)

Fig. 21 *b*.

American-born Japanese and Japan-born Japanese.

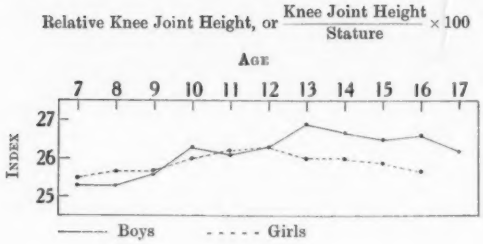


Fig. 22 *a*.

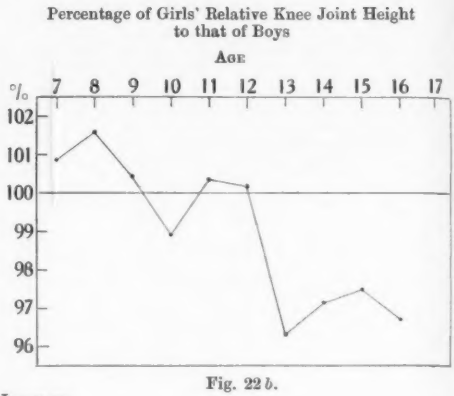


Fig. 22 *b*.

American-born Japanese.

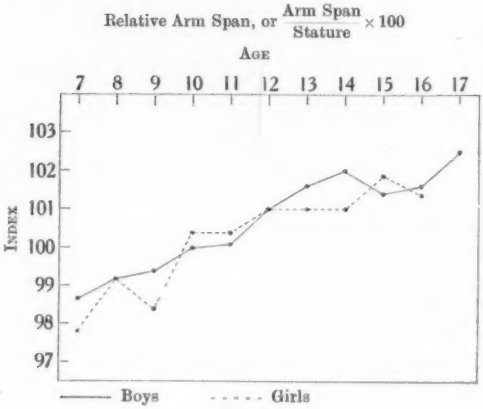


Fig. 23 *a*.

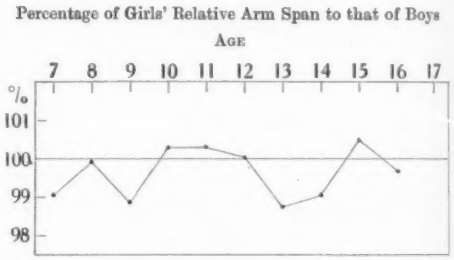


Fig. 23 *b*.

American-born Japanese.

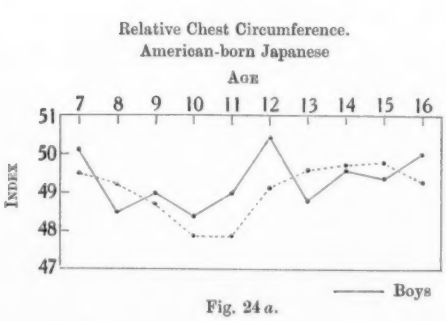


Fig. 24 *a*.

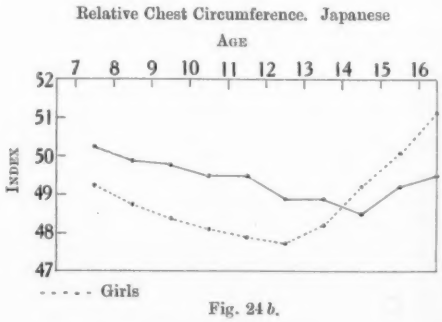


Fig. 24 *b*.

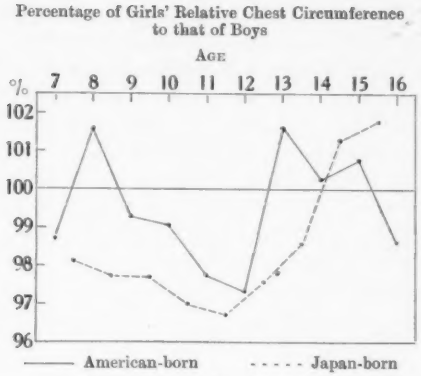


Fig. 24 *c*.

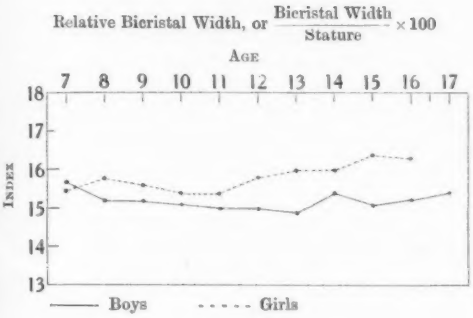


Fig. 25 *a*.

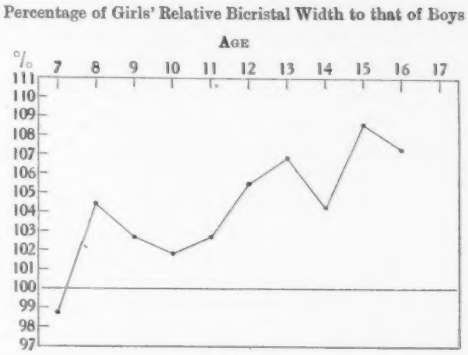


Fig. 25 *b*.

American-born Japanese.

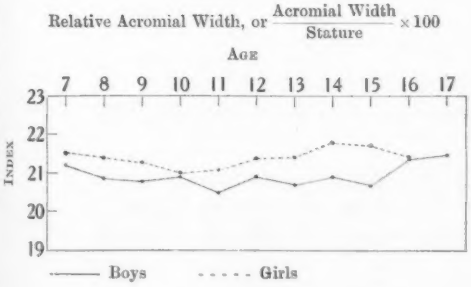


Fig. 26 *a*.

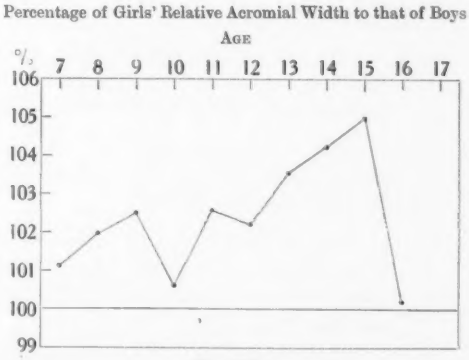


Fig. 26 *b*.

American-born Japanese.

The curves of the percentages of girls' stature and weight to those of boys (Tables VI, VII, Figs. 17, 18) run the same general course, whether they be for Americans, Japanese, American-born Japanese or Europeans. The curves show the pre-puberty upheaval over the boys' 100 % line between the ages of 11 and 14 years in stature, and between 12 and 15 years in weight. It is noticeable that the curve of the American-born Japanese is slightly irregular on account of the smaller number of observations. Figures for Americans as well as for Japanese are based on measurements of millions; those of Germans are thought to be equally large; but those of American-born Japanese are only a little over a thousand. That the curves of the American-born Japanese girls show a sudden and rapid drop after reaching a maximum at 12 years, may be explained as the general poor physical condition of the girls of 13 years or more. After making measurements of the same group of children a number of times in the future, I may be able to offer a better explanation. The curves of the German girls' acceleration tend to be delayed for about 2 years. This is probably caused by the delay in puberty on account of higher latitude and colder climate.

The relative sitting height of American-born Japanese is remarkably low when compared with that of Japan-borns. It is known that this index is highest at birth and goes down with age until it reaches the lowest level at a definite age and then again goes up a little. Bean (22) calls this index a "skeletal index" and finds it inverse in stature (that is smaller in taller people than in shorter). He found Whites and Filipinos to have about the same index and Negroes smaller. The skeletal index, according to Bean, reaches the minimal level between ages 10 to 16 and 9 to 14 in American boys and girls respectively, and between 8 to 16 and 8 to 12 in German-American boys and girls respectively. The minimal level for American-born Japanese boys is reached at 15, and the same for Japan-born Japanese boys at 14, the minimum for girls being reached a year or two earlier in either case. (Table VIII and Fig. 19 *a*.)

The percentage of girls' relative sitting height is nearly at 100 % level at 7 years but it rises gradually till about 15 years of age, to the level between 103 and 104 in both the American-born and Japan-born Japanese. (Table IX and Fig. 19 *b*.)

The relative height of anterior superior iliac spine goes up from about 54 at 7 years to around 56 at 10 years of age, from then on there is no marked change, only the girls' figures drop somewhat after 13 years of age.

The percentage of girls' relative height of anterior superior iliac spine to that of boys is nearly at 100 % mark up to 11 years, then it drops off gradually. (See Table IX and Fig. 20 *b*.)

The relative leg length of American-born Japanese is, in general, greater than that of Japan-born by 4·0 to 4·5, the general tendency of both being a gradual climb with age up to 10 or 11 years, after which it stays rather stationary. (Table X and Fig. 21 *a*.)

The percentage of girls' relative leg length to that of boys runs just about the

TABLE IX. *The Percentage of Girls' Relative Measures to those of Boys.*

Age	Sitting Height	Iliac Spine Height	Leg Length	Knee Joint Height	Arm Span	Chest Circumf.	Cristal Width	Acromial Width
			American-born Japanese Children					
7	99·29	100·72	100·74	100·94	99·04	98·71	98·81	101·14
8	100·82	99·91	99·87	101·59	99·99	101·59	104·35	101·99
9	100·35	99·78	99·74	100·43	98·94	99·23	102·75	102·47
10	99·78	99·82	100·36	98·88	100·31	99·02	101·93	100·55
11	100·51	100·11	100·77	100·37	100·30	97·66	102·83	102·53
12	100·32	99·66	100·24	100·21	100·04	97·28	105·49	102·23
13	101·67	98·94	98·89	96·32	98·76	101·59	106·85	103·54
14	103·13	98·46	99·48	97·14	99·03	100·23	104·15	104·15
15	104·18	99·42	100·38	97·48	100·50	100·79	108·57	104·99
16	103·47	98·23	97·96	96·66	99·72	98·55	107·19	100·16
			Japan-born Japanese Children					
$7\frac{1}{2}$	100·16	—	99·2	—	—	98·1	—	—
$8\frac{1}{2}$	100·10	—	99·2	—	—	97·8	—	—
$9\frac{1}{2}$	100·30	—	100·6	—	—	97·7	—	—
$10\frac{1}{2}$	100·70	—	100·6	—	—	97·0	—	—
$11\frac{1}{2}$	100·79	—	100·4	—	—	96·8	—	—
$12\frac{1}{2}$	101·21	—	99·6	—	—	97·6	—	—
$13\frac{1}{2}$	101·92	—	99·8	—	—	98·6	—	—
$14\frac{1}{2}$	103·15	—	99·8	—	—	101·3	—	—
$15\frac{1}{2}$	102·97	—	—	—	—	101·8	—	—
$16\frac{1}{2}$	102·62	—	—	—	—	—	—	—
	These are from the figures of Drs Tsurumi, Nakadate, Yagi and Toyoda combined		These are from the figures of Drs Mishima and Minayoshi combined			These are from the figures published by the Ministry of Education, Japan		

TABLE X. *Relative Leg Length of American-born Japanese Children compared with that of Japanese Children in Japan.*

Age	American-born Japanese		Japanese in Japan. Measurements of Drs Mishima and Minayoshi combined	
	Boys	Girls	Boys	Girls
7	52·3	52·7	49·2	48·8
8	53·4	53·4	49·3	48·9
9	53·7	53·6	49·5	49·8
10	54·6	54·8	49·6	49·9
11	54·7	55·1	49·7	49·9
12	54·6	54·7	50·0	49·8
13	54·9	54·3	50·0	49·9
14	54·5	54·2	50·5	49·9
15	54·2	54·4	—	—
16	54·3	53·2	—	—

same course in American-born as in Japan-born Japanese. The girls excel boys between 9½ and 12 years of age. (See Table IX and Fig. 21 *b*.)

The relative knee joint height of American-born Japanese shows a slight increase with age. The percentage of girls' relative knee joint height to that of boys has a downward tendency, being above the boys' level at 7 years and dropping down to about 96·5% at 16 years of age. (See Table IX and Fig. 22 *b*.)

The relative arm span of American-born Japanese climbs up from 98 to 102 in the course of years 7 to 17. The values for boys and girls are nearly the same. (See Table IX and Figs. 23 *a*—*b*.)

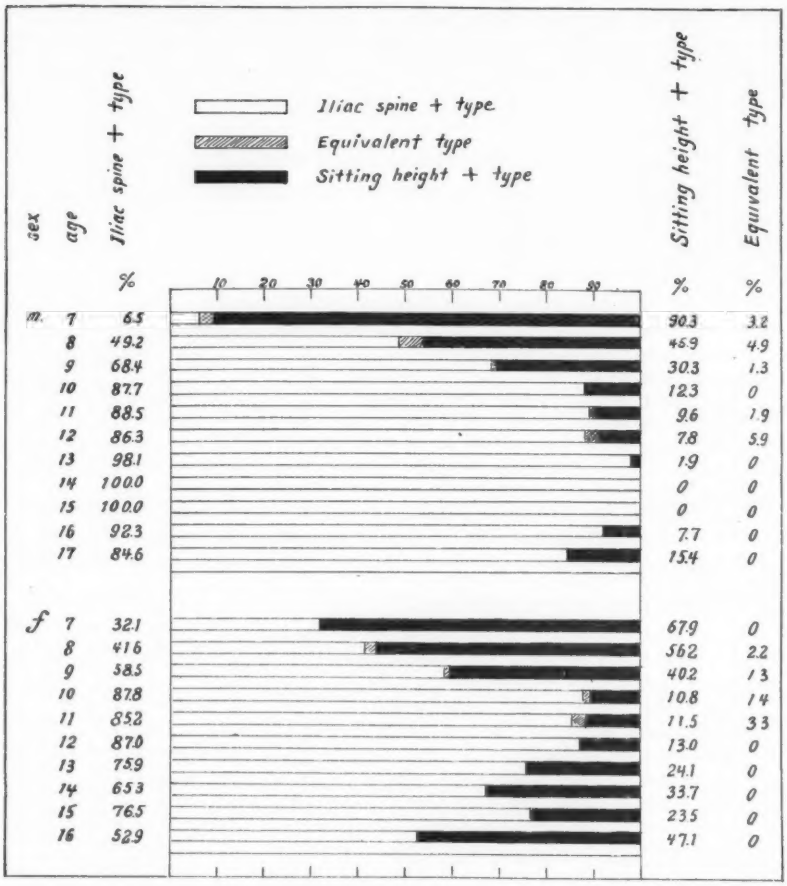


Fig. 27.

The relative chest circumference takes a course first down and then up, the lowest level being at 10 to 11 years in the case of American-born and at about the 13th year in case of Japan-born Japanese. There seems to be a greater difference between boys' and girls' figures in Japan-born Japanese than those of American-born Japanese. (Table IX and Fig. 24 *c*.)

The relative bicristal width and the relative acromial width of American-born Japanese are kept almost at the same level all the way through the ages 7 to 17. The percentage of girls' relative values to those of boys is always above the boys' line, and the general tendency is a steady climb. (Table IX and Figs. 25 *b*, 26 *b*.)

TABLE XI.

American Native White Stock (Third Generation Native-born).

Collins and Clark, *U.S. Public Health Reports* 44; 18, 1059, 1929.

Age	Standing Height (cm.)		Sitting Height (cm.)		Chest Circumf. (cm.)		Weight (kg.)	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
7	119·5	118·5	63·8	63·3	59·3	58·1	22·6	22·1
8	125·1	124·3	66·0	65·5	60·9	59·1	25·1	24·3
9	130·2	129·4	68·0	67·5	62·5	61·3	27·6	27·1
10	135·1	134·5	69·8	69·5	64·3	63·2	30·4	29·8
11	139·9	140·1*	71·5	71·9*	66·2	65·8	33·3	33·7*
12	144·4	146·0*	73·3	74·4*	68·1	68·2*	36·4	37·8*
13	150·0	152·0	75·5	77·6*	70·7	71·1*	40·5	43·0*
14	155·5	155·5	78·2	79·6*	73·6	73·3	45·1	46·8*
15	158·9	157·0	79·8	80·8*	74·4	73·7	47·3	49·5*
Actual Mean Annual Increment								
7— 8	5·6	5·8*	2·2	2·2	1·6	1·3	2·5	2·2
8— 9	5·1	5·1	2·0	2·0	1·6	1·9*	2·5	2·8*
9—10	4·9	5·1*	1·8	2·0*	1·8	1·9*	2·8	2·7
10—11	4·8	5·6*	1·7	2·4*	1·9	2·6*	2·9	3·9*
11—12	4·5	5·9*	1·8	2·5*	1·9	2·4*	3·1	4·1*
12—13	5·6	6·0*	2·2	3·2*	2·6	2·9*	4·1	5·2*
13—14	5·6	3·5	2·7	2·0	2·9	2·2	4·6	3·8
14—15	3·4	1·5	1·6	1·2	0·8	0·4	2·2	2·7*
Percentage Increase per year								
7— 8	4·72	4·87*	3·47	3·55*	2·84	2·25	11·03	9·98
8— 9	4·08	4·11*	3·05	2·97	2·53	3·28*	9·96	11·57*
9—10	3·76	3·94*	2·65	2·97*	2·85	3·05*	10·05	10·27*
10—11	3·54	4·20*	2·45	3·53*	2·98	4·17*	9·77	13·02*
11—12	3·19	4·19*	2·48	3·51*	2·98	3·67*	9·09	12·17*
12—13	3·92	4·13*	3·11	4·23*	3·65	4·25*	11·51	13·60*
13—14	3·67	2·28	3·55	2·58	4·18	2·98	11·37	8·83

* Those marked with an asterisk are instances in which girls excel boys.

The sitting height and the anterior superior spine height are nearly the same in some children, but not in all. A child may be either one of these, i.e., these measurements are nearly equal, or the iliac spine height is greater, or the sitting height is greater. Whether these types, if they may be so called, have something to do with body types of children, is yet to be determined. For the present, I will just divide the boys and girls of the American-born Japanese into these three types, viz.:

1. Iliac spine plus type. (Where the iliac spine height is greater than the sitting height.)

2. Equivalent type. (Where they are equal within 0·5 cm.)

3. Sitting height plus type. (Where the sitting height is greater than the iliac spine height.)

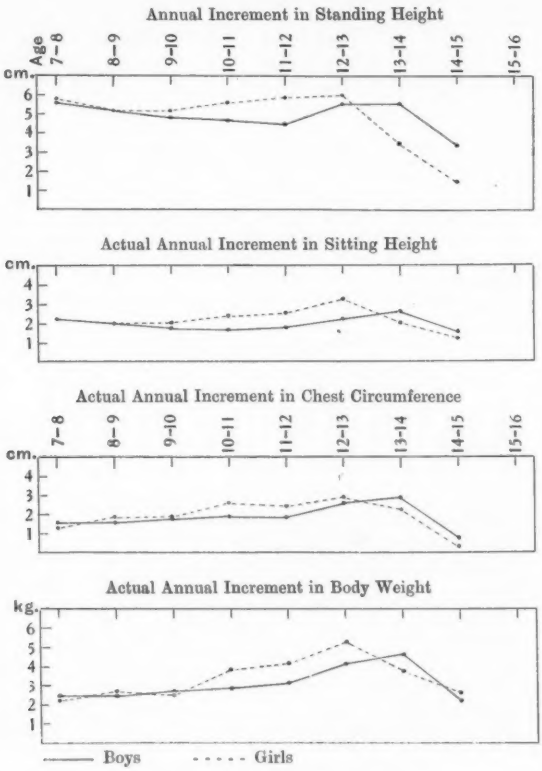


Fig. 28. American Native White Stock.

TABLE XII. *American-born Japanese (Second Generation).*

Age	Standing Height cms.		Sitting Height cms.		Chest Circumf. cms.		Weight kg.	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
7	115·5	115·6*	64·3	63·9	57·9	57·2	22·3	21·1
8	121·3	119·4	66·1	65·6	55·8	58·8*	23·9	22·6
9	126·2	124·1	68·2	67·3	61·0	60·4	26·5	24·6
10	131·6	130·4	70·6	69·8	63·7	62·5	29·7	28·2
11	135·9	135·4	72·3	72·4*	66·6	64·8	32·8	30·2
12	139·7	143·2*	74·2	76·3*	70·5	70·3	34·8	38·1*
13	147·3	146·0	77·7	78·3*	71·9	72·4*	39·5	39·8*
14	155·6	149·4	81·9	81·1	77·1	74·2	46·3	43·1
15	161·3	150·4	83·9	81·5	79·7	74·9	50·0	43·4
Actual Mean Annual Increment								
7— 8	5·8	3·8	1·8	1·7	−2·1	1·6	1·6	1·5
8— 9	4·9	4·7	2·1	1·7	5·2	1·6	2·6	2·0
9—10	5·4	6·3*	2·4	2·5*	2·7	2·1	3·2	3·6*
10—11	4·3	5·0*	1·7	2·6*	2·9	2·3	3·1	2·0
11—12	3·8	7·8*	1·9	3·9*	3·9	5·5*	2·0	7·9*
12—13	7·6	2·8	3·5	2·0	1·4	2·1*	4·7	1·7
13—14	8·3	3·4	4·2	2·8	5·2	1·8	6·8	3·3
14—15	5·7	1·0	2·0	0·4	2·6	0·7	3·7	0·3
Percentage Increase per year								
7— 8	5·03	3·23	2·80	2·66	—	2·80	7·18	7·11
8— 9	4·04	3·94	3·18	2·59	9·32	2·72	10·88	8·85
9—10	4·28	5·08*	3·52	3·72*	4·43	3·48	12·07	14·63*
10—11	3·27	3·83*	2·41	3·73*	4·55	3·68	10·54	7·09
11—12	2·79	5·76*	2·63	5·39*	5·86	8·59*	6·09	26·15*
12—13	5·44	1·96	4·72	2·62	1·99	2·99*	13·51	4·46
13—14	5·64	2·33	5·41	3·58	7·23	2·49	17·22	8·29
14—15	3·66	0·67	2·44	0·49	3·37	0·94	7·99	0·70

TABLE XIII. *Actual Mean Annual Increment of other Measures than those listed in the last Table.*

Age	Anterior Iliac Sp. Height (cm.)		Leg Length (cm.)		(cm.) Knee Joint Height (cm.)		Arm Span (cm.)		Bicristal Width (cm.)		Acromial Width (kg.)	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
7— 8	4·4	2·8	4·4	2·8	−1·8	1·2	6·3	5·4	0·3	1·0*	0·9	0·7
8— 9	3·0	2·8	2·9	2·6	1·6	1·2	5·2	3·7	0·8	0·5	0·8	0·9*
9—10	4·4	4·9*	3·8	4·8*	2·3	2·0	6·2	8·8*	0·7	0·7	1·3	1·0
10—11	2·6	3·2*	2·8	3·1*	0·9	1·6*	4·4	5·1*	0·5	0·8*	0·4	1·1*
11—12	2·4	4·3*	2·4	4·1*	1·2	2·2*	5·0	8·7*	0·5	1·7*	1·3	2·1*
12—13	4·2	0·9	3·8	0·9	3·0	0·2	8·2	2·8	1·1	0·7	1·3	0·7
13—14	4·9	1·7	4·5	1·5	1·9	0·9	9·4	3·4	1·9	0·6	2·0	1·2
14—15	2·6	0·8	2·2	0·6	1·2	0·1	4·9	2·4	0·4	0·7*	0·8	0·1
15—16	0·3	0·0	0·4	0·0	0·1	0·1	0·5	0·6*	0·3	0·2	1·2	−0·1

* Those marked with an asterisk are instances in which girls excel boys.

The percentage of the number of each type at each age and sex is given in Fig. 27. So far I have not seen reports of this nature by other investigators. I cannot therefore make any comparison or draw any conclusion. One may see,

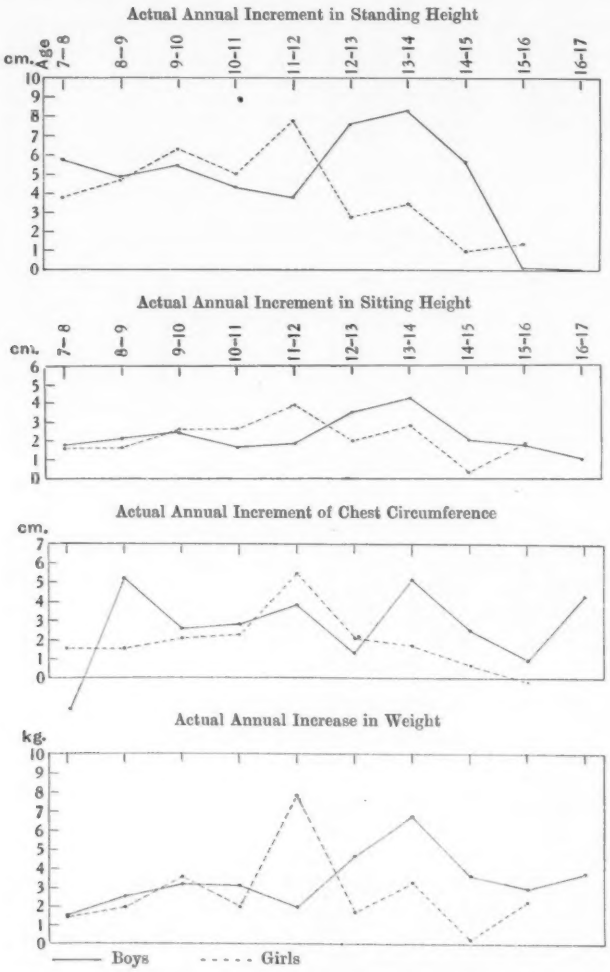


Fig. 29. American-born Japanese Boys and Girls.

however, that the maximum of iliac spine plus type in the boys is reached at ages 13 to 15, while the maximum of the same type for the girls is reached at ages 10 to 12, though less pronounced in degree. The equivalent type seems to have no significance.

Collins and Clark (21) published valuable data on American native white stock (third generation native born children), for which they calculated actual mean annual increment and annual percentage of increase, in stature, sitting height, chest circumference and body weight. (See Table XI and Fig. 28.) Girls seem to have greater actual annual increment than boys during the ages between 10 and 13. These figures are based on measurements of 28,674 children. My figures on the

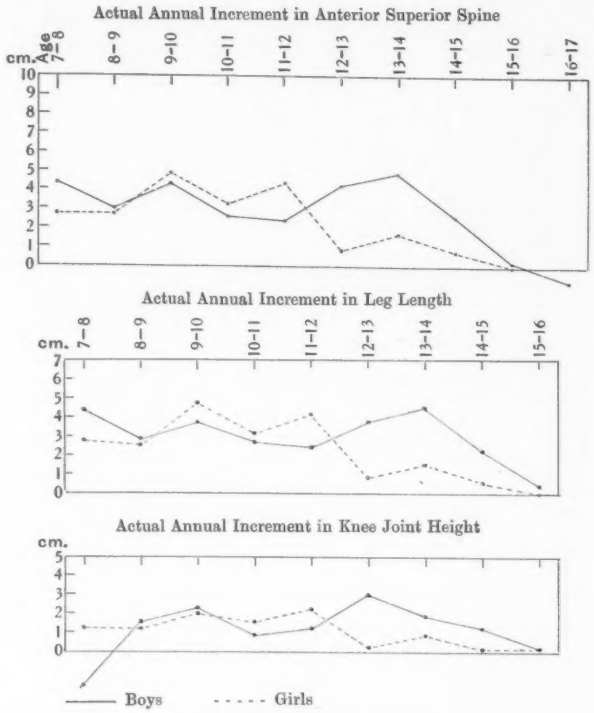


Fig. 30. American-born Japanese.

measurement of a little over 1,000 children may not be worth much, but I publish them here (Tables XII and XIII, Figs. 29, 30 and 31), simply to show that at least the girls' annual increment is decidedly greater than that of boys in all measurements, at the age of 12.

In conclusion, I have computed the index of measurements of the American-born Japanese, taking the figures of the Japan-born Japanese as a standard, thus:

$$\frac{\text{Measurement of American-born} \times 100}{\text{Measurement of Japan-born}}$$

and found that, in stature, the American-born are higher by 6 to 7 %, although girls drop to 3·5 % at the age of 14½. The relative sitting height of American-born is below that of Japan-born by 4 % up to 11½ years, after which it climbs up to about − 2 %. The relative leg length of American-born is more than 6 % greater at 7 years than that of Japan-born, and rises to 10 % at 11½ years.

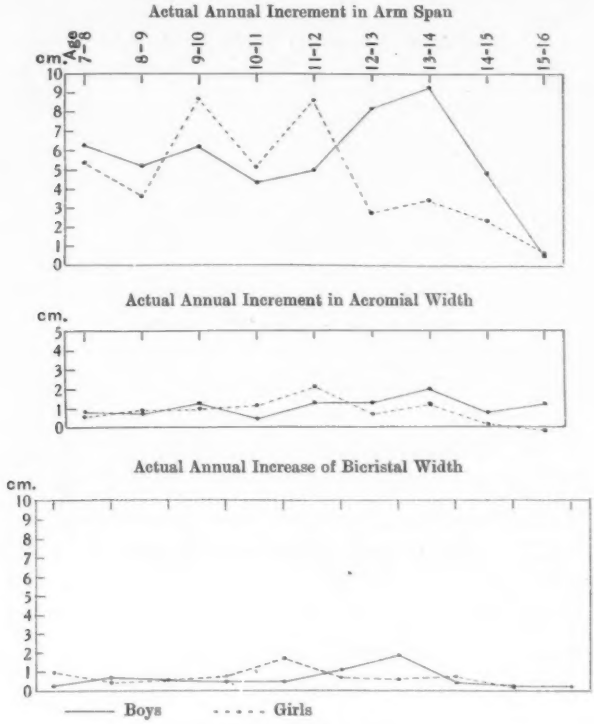


Fig. 31. American-born Japanese

The chest circumference of American-born climbs from 1 % plus to 9 % above that of Japan-born at the age of 14½ in case of boys, while the girls' figures are generally higher, i.e., rise from 7 % at 7 years to over 10 % at 12½ years, then drop to 5 % at 14½ years.

The body weight of American-born ranges from 19 to 26 % above that of Japan-born in the case of boys. The girls' figures are lower than this by 2 to 3 %, but the lowest level at 14½ years is still over 9 % above that of Japan-born. (Table XIV and Figs. 32, 33.)

Taken as a whole, the difference in body build of the American-born Japanese

TABLE XIV.

Index, Values of American-born Japanese, with Values of Japan-born Japanese Children taken as a Standard.

The figures of the American-born Japanese are increased by ½ annual increment in order to raise to the age of the Japanese which is year+½.

Age	Stature	Rel. Sit.	Rel. Leg	Chest Circ.	Weight
BOYS					
7½	106·00	96·87	106·30	101·25	119·69
8½	106·63	95·28	108·32	100·86	119·43
9½	106·88	95·75	108·48	104·00	121·64
10½	107·13	95·71	110·08	105·50	124·70
11½	106·74	96·03	110·06	106·42	123·36
12½	107·17	96·90	109·45	108·70	123·59
13½	108·22	97·59	112·35	108·92	126·18
14½	107·54	98·14	109·00	109·50	122·65
Average	7·04	−3·47	9·26	5·64	22·66
GIRLS					
7½	106·82	96·01	108·00	107·01	118·38
8½	106·38	95·98	109·20	106·81	116·26
9½	106·89	95·93	107·63	106·77	118·92
10½	107·20	94·86	109·82	107·24	119·67
11½	107·90	95·71	110·42	109·38	125·28
12½	107·03	96·04	109·84	110·70	125·48
13½	105·20	97·28	108·82	108·44	117·90
14½	103·52	98·19	108·62	104·77	109·34
Average	6·37	−3·75	9·04	7·64	18·90
Average for both sexes	6·7	−3·6	9·15	6·64	20·8

(the so-called second generation Japanese, as their parents are of Japan birth) from that of Japan-born children is, in round numbers, 7 % higher in stature, 4 % shorter in relative sitting height, 9 % greater in relative leg length, 7 % greater in chest circumference, and 20 % heavier in weight.

Both parents of these children were full-blooded Japanese born in Japan in all instances. There were only a few pupils of mixed Japanese and Caucasian parentage in the two schools where measurements were made, and they were not included in the present investigation. Even here in America, the intermarriage of Japanese with other races is very rare, as the Japanese community is fairly large and Eurasian children are unusual.

The children in this investigation are certainly growing faster, that is taller and heavier, age for age, in comparison with children in Japan. It seems very likely that they will grow up to be taller and heavier adults than native born Japanese, inferring from the height and weight attained by these children at 16 or 17 years of age, which were already greater than those of adult Japanese.

A few years ago, I measured a hundred young Japanese of American birth whose ages were 19 years and upwards, and found them to surpass their fathers in height by an average of more than 3 inches. When I have measured the children of the present study annually for a few years, I expect to be able to report more definitely what their ultimate stature and weight will be.

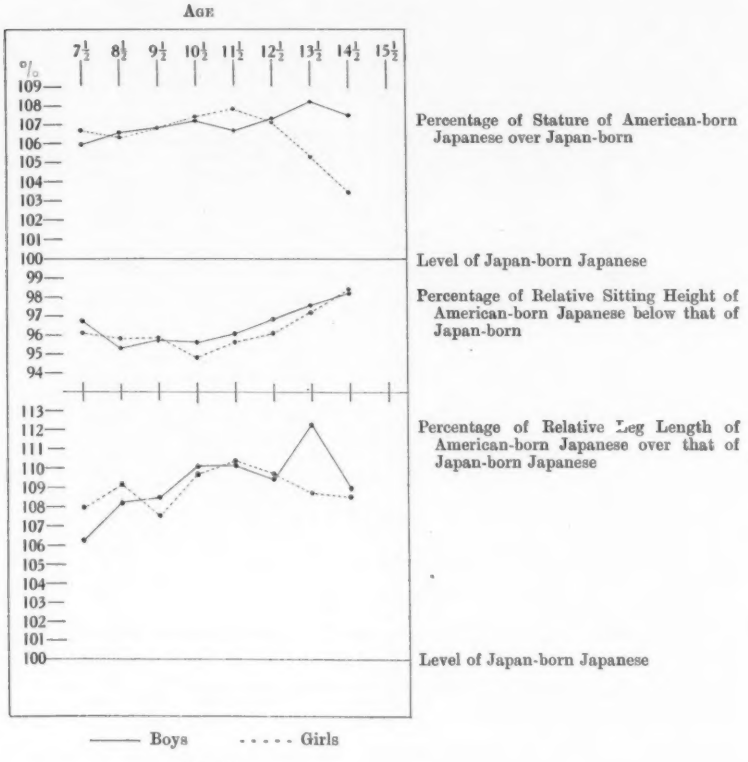


Fig. 32.

Prof. K. Pearson tells me in a personal communication of his experience with two tall Japanese, who said their stature was quite usual in the case of the Japanese island from which they came. I am not aware of any island in Japan which is noted for a higher stature of its inhabitants. As far as I know, the people of Japan cannot be divided into racial groups, with the exception of the Ainus in the extreme north. The latter are not included in the present study. Parents of the children in this work come from all parts of Japan, and they may be regarded as representing the pure Japanese race. The Japanese Government measurements were performed under the supervision of Dr Yoshida, who follows Martin's method as I do. The

Government measured the school children from all over Japan, that is Japan proper, where no different races figured, annually for a number of years. Therefore I think I am justified in making a comparison between my measurements and those of the Japanese Government.

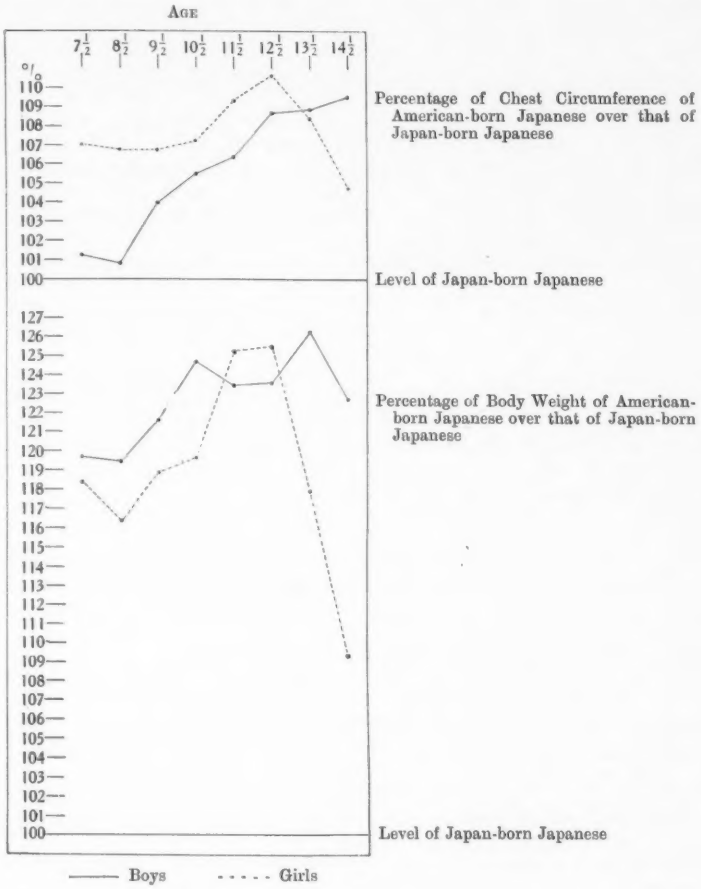

AGE
7½ 8½ 9½ 10½ 11½ 12½ 13½ 14½
%
110 109 108 107 106 105 104 103 102 101 100
Percentage of Chest Circumference of American-born Japanese over that of Japan-born Japanese
Level of Japan-born Japanese
127 126 125 124 123 122 121 120 119 118 117 116 115 114 113 112 111 110 109 108 107 106 105 104 103 102 101 100
Percentage of Body Weight of American-born Japanese over that of Japan-born Japanese
Level of Japan-born Japanese
Boys
Girls

Fig. 33.

No appreciable difference is shown in stature and weight between social and labouring classes in Japan. Of course, the weight and stature must be greater in better nourished children of well-to-do families when compared with those of paupers. In any case, my figures as well as those of the Japanese Government measurements cover all classes.

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METHODS OF STATISTICAL ANALYSIS APPROPRIATE FOR k SAMPLES OF TWO VARIABLES.

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I. Introduction.

(1) *The Testing of Statistical Hypotheses.* Statistical theory which is not purely descriptive is largely concerned with the development of tools which will assist in the determination from observed events of the probable nature of the underlying cause system that controls them. The measured characteristics of quality vary from unit to unit, and the statistical technique is required to analyse this variation and covariation to break it into parts with which may be associated assignable causes, to test and compare alternative hypotheses, and to express the resulting conclusions in terms of measures of probability. It will be found that some of the most recent generalisation of theory has resulted from an attempt to provide critical tests of increasingly complex hypotheses. We may trace the development through a chain of questionings: Is it likely, (a) that this sample has been drawn from a specified population, P; (b) that these two samples have come from a common but unspecified population; (c) that these k samples have come from a common but unspecified population? Again the population P may be (d) completely specified or, (e) only partly specified, e.g. its mean is given but not its standard deviation; or when there are a number of samples we may allow the means in the sampled population to be different and question whether the standard deviations are the same. Another line of advance is from (f) problems dealing only with a single variable, to (g) those in which there may be a number of correlated variables.

Now we may frankly admit that in so far as the technique is to be used in handling data broken up into small groups, the recent theoretical developments assume normal variation. But to place the procedure for testing statistical hypotheses on a firm logical basis under one set of simplified conditions, is in itself an

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achievement of some value, and perhaps the most practical line of advance is the following:

(*a*) To establish what we may term "normal theory."

(*b*) To study in a more systematic way than has been attempted the extent of departure from normality met with in different fields of application.

(*c*) To examine how rapidly normal theory tests become inefficient as the form of variation and covariation departs from the normal, and to determine the nature of the errors in judgment that will arise if these tests are still used.

(2) *The Analysis of Variance.* R. A. Fisher's methods of Analysis of Variance may be regarded from the following viewpoint:

(*a*) In any given problem it will generally be possible to specify certain factors which may be the cause of part of the variation, while there will be a residual part which, in the state of knowledge at the moment, must be regarded as due to unidentifiable or chance causes.

(*b*) An experiment may be designed to test whether a certain factor is operative or not; for example:

(i) Do differences in manurial treatment affect the yield of some variety of cereal?

(ii) Does modification in the production process alter the quality of output of some manufactured product?

(*c*) At the same time, in addition to these factors whose influence is under investigation, there may be other assignable causes of variation inevitably present, the effect of which would be obscuring were it not eliminated. Thus, for example, in the illustrations given above, there might be

(i) Variation due to changing soil fertility.

(ii) Variation due to differences in the skill of operatives or in the state of wear of machines.

(*d*) It follows that it is often possible to regard the variation in a character x as made up of parts due to different assignable causes A, B, C, ..., and of a residual part which, for the time being, we must attribute to chance causes. This may be expressed as follows:

$$x_{t.\,u,v,w,\ldots} = a_u + b_v + c_w + \ldots + X_{t.\,u,v,w,\ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),*$$

where $x_{t.\,u,v,w,\ldots}$ is the character of the t-th individual of a group of $n_{u,v,w,\ldots}$, all of which receive the same contribution a_u from the A factor, the same contribution b_v from the B factor and so on. $X_{t.\,u,v,w,\ldots}$ represents the residual term. In so far as the causes of variation are assignable, this grouping is possible.

* [The reader must bear in mind that for (1) to be true the effect of the causes A, B, C, ... on the character x must be *additive*. For example, the real effect of A and B might lead to a ratio a_u/b_v in the expression for $x_{t.\,u,v,w,\ldots}$, in which case the assumption of an additive relation would involve the influence of A, B, C, ... appearing in $X_{t.\,u,v,w,\ldots}$. Ed.]

(*e*) The technique of analysis consists in arranging the data so as to test separately for the presence of an A factor, or a B factor, etc. as desired. This is effected by obtaining in each case two estimates of the unknown variance, σ^2, of the residuals X, which would differ only through chance fluctuations if changes in the particular factor had no influence on variation within the limits covered by the experiment.

This method of analysis is based upon the assumptions that the residuals X, (*a*) are normally distributed, and (*b*) have the same standard deviation, σ, whatever be the values of the terms a, b, c, etc., that is to say, for all combinations of the assignable causes. We may be justified in accepting this to be the true position in very many practical cases, but it should be recognised that the method outlined above does not put these assumptions to the test. There are indeed a number of problems in which (*b*) is not true and where the discovery of significant differences, from group to group, in the variation among the individuals, X, may lead to the identification of further assignable causes of variation. Such has been found to be the case, for example, in the analysis of variation in quality of articles in mass-production industry. Further, we may be concerned not only with a single variable x, but with a number of correlated variables x, y, z, ..., and we may then need to examine the stability from group to group of the covariation as well as the variation among residuals X, Y, Z, ...

(3) *An Illustration of the Problem.* The purpose of this paper is to develop certain methods recently suggested for dealing with these problems*. We shall treat here only the case of two correlated variables x and y, and shall suppose that the observations have been divided into k samples or groups. The problem will be to test whether these groups can be differentiated owing to significant differences either in the average values of x and y, or in the variation and covariation of the residuals within the groups. The choice of suitable criteria, i.e. of the tests to be applied, has been based upon the use of the principle of likelihood as suggested by J. Neyman and E. S. Pearson. More recently† these writers have suggested a more fundamental method of determining the most efficient test of a statistical hypothesis; this method of choice has been shown in many cases to be identical with the method of likelihood, but in the particular problems we are now considering the correspondence has not yet been established.

The following example, which is treated more fully below, will indicate the nature of the problem. In certain cases of manufacture tests of quality are destructive. Such, for example, is a test for breaking strength; it is therefore important to find an alternative correlated measure which may be used in its place in routine testing. In dealing with metal products a measure of hardness, based on a test which is not destructive, is sometimes used as an index of tensile strength. If,

* For the case of a single variable: see J. Neyman and E. S. Pearson: *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1930 and 1931. For the case of many variables, see S. S. Wilks: *Biometrika*, Vol. XXIV (1932), pp. 471—494.

† *Phil. Trans. of the Royal Soc.*, Series A, Vol. 231, pp. 289—337.

however, we are to predict strength from hardness, using the correlation method, it is essential that the degree of relationship between the two qualities should remain stable. It must not change from one plant to another or from one month to the next; in other words a preliminary research investigation should not only be concerned with changes in average strength and hardness which can be attributed to assignable causes, but also with the stability of the covariation among the residuals X and Y. Table I shows a preliminary statistical analysis of 60 pairs of test results made on a certain aluminium die-casting, divided into 5 groups of 12 pairs. Within a group the assignable causes of variation are believed to be constant, but it is necessary to analyze not only the figures in the 2nd and 4th columns, but also those in the 3rd, 5th and 6th.

TABLE I.

Data for Aluminium Die-Castings. (Samples of 12 observations.)*

Sample No.	Tensile Strength (10^3 lb. per sq. in.)		Hardness (Rockwell's E)		Coefficient of Correlation
	Mean	Standard Deviation	Mean	Standard Deviation	
1	33·399	2·565	68·49	10·19	0·683
2	28·216	4·318	68·02	14·49	0·876
3	30·313	2·188	66·57	10·17	0·714
4	33·150	3·954	76·12	11·08	0·715
5	34·269	2·715	69·92	9·88	0·805

II. Derivation of the Criteria.

(4) We shall suppose that each of k samples, $\Sigma_1, \Sigma_2, \ldots \Sigma_k$, of two variables x and y has been drawn from some normal population. Let π_t be the population from which Σ_t has been drawn and let the means of x and y in π_t be a_t and b_t, the standard deviations σ_{xt} and σ_{yt} and the correlation coefficient $\rho_t\,(t = 1, 2, \ldots k)$. Thus, the distribution law of π_t will be

$$\frac{1}{2\pi\sigma_{xt}\sigma_{yt}\sqrt{1-\rho_t^2}}\, e^{-\frac{1}{2(1-\rho_t^2)}\left[\frac{(x-a_t)^2}{\sigma_{xt}^2}+\frac{(y-b_t)^2}{\sigma_{yt}^2}-\frac{2\rho_t(x-a_t)(y-b_t)}{\sigma_{xt}\sigma_{yt}}\right]} \quad \ldots\ldots(2).$$

Therefore, the probability of the joint occurrence of the samples Σ_t from their respective populations $\pi_t\,(t = 1, 2, \ldots k)$, with values of x and y falling in the intervals $x_{ta} \pm dx_{ta}$, $y_{ta} \pm dy_{ta}$ $(a = 1, 2, \ldots n_t,\ t = 1, 2, \ldots k)$ will be given, except for infinitesimals of higher order than dx_{ta} and dy_{ta}, by

$$C = \prod_{t=1}^{k}\left(\frac{1}{2\pi\sigma_{xt}\sigma_{yt}\sqrt{1-\rho_t^2}}\right)^{n_t} e^{-\theta}\, dX\, dY \quad \ldots\ldots(3),$$

* The data are taken from W. A. Shewhart's *Economic Control of Manufactured Product*, Macmillan 1931. Although they would hardly be adequate for a research investigation in practice, they are suggestive and provide a good illustration of method.

in which

$$\theta = \sum_{t=1}^{k} n_t \left[\frac{s_{xt}^2 + (\bar{x}_t - a_t)^2}{2\sigma_{xt}^2(1-\rho_t^2)} + \frac{s_{yt}^2 + (\bar{y}_t - b_t)^2}{2\sigma_{yt}^2(1-\rho_t^2)} - \frac{2\rho_t[s_{xt}s_{yt}r_t + (\bar{x}_t - a_t)(\bar{y}_t - b_t)]}{2\sigma_{xt}\sigma_{yt}(1-\rho_t^2)} \right] \quad (4),$$

where $\bar{x}_t$ and $\bar{y}_t$ are the means, s_{xt} and s_{yt} the standard deviations *, r_t the correlation coefficient of x and y, and n_t the number of individuals in the sample Σ_t, and

$$dX\,dY = \prod_{t=1}^{k} \prod_{\alpha=1}^{n_t} dx_{t\alpha}\,dy_{t\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

We shall now consider the derivation of criteria for testing the following three hypotheses concerning the populations π_t:

(i) The hypothesis H that the populations π_t are identical, that is, that

$$\sigma_{xt} = \sigma_x, \quad \sigma_{yt} = \sigma_y, \quad \rho_t = \rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$a_t = a, \quad b_t = b \qquad (t = 1, 2, \ldots k) \ldots\ldots\ldots\ldots (7).$$

(ii) The hypothesis H_1 that the samples have come from populations with the same set of variances and correlations but having means with any differing values whatever, that is, that (6) is true whatever may be the values of the means a_t and b_t.

(iii) The hypothesis H_2 that the samples are from populations in which (7) is true, when it is assumed that (6) is true.

These are generalisations to two variables of the three hypotheses considered by Neyman and Pearson† for the case of k samples of a single variable; or they may be regarded as special cases of the more general problem whose solution has been considered more recently by Wilks‡. Thus, it will only be necessary to indicate briefly the steps involved in applying the method of likelihood to determine criteria appropriate in testing H, H_1 and H_2. In each case we must fix:

(a) The class Ω of admissible sets of populations π_t ($t = 1, 2, \ldots k$) from one set of which the set of samples Σ_t is assumed to have been drawn.

(b) The subclass ω of Ω to which the set π_t must belong if the hypothesis tested be true.

Then we must find the maximum of C in (3) for variations of the population parameters under the assumption that the set π_t is (i) a member of Ω; call this $C(\Omega\ \max)$; and (ii) a member of ω; this we call $C(\omega\ \max)$. Then the expression for the likelihood of the composite hypothesis H has been defined to be

$$\lambda_H = \frac{C(\omega\ \max)}{C(\Omega\ \max)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Let us consider this λ-criterion for each of the hypotheses H, H_1 and H_2.

* Here and throughout the paper the standard deviation in a sample of n will be defined by the relation $ns^2 = \Sigma(x - \bar{x})^2$.

† *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1931.

‡ *Loc. cit.*

(i) Criterion for H. We find that $C(\Omega \text{ max})$ occurs when

$$a_t = \bar{x}_t, \quad b_t = \bar{y}_t \qquad \text{.....(9),}$$

$$\sigma_{xt} = s_{xt}, \quad \sigma_{yt} = s_{yt}, \quad \rho_t = r_t \qquad (t = 1, 2, \ldots k) \quad (10),$$

and $C(\omega \text{ max})$ occurs when

$$a = \bar{x}_0, \quad b = \bar{y}_0 \qquad \text{.....(11),}$$

$$\sigma_x^2 = v_{110} = v_{11a} + v_{11m}, \quad \sigma_y^2 = v_{220} + v_{22m}, \quad \sigma_x \sigma_y \rho = v_{120} = v_{12a} + v_{12m} \quad \text{...(12),}$$

where

$$\bar{x}_0 = \frac{1}{N} \sum_{t=1}^{k} n_t \bar{x}_t, \quad \bar{y} = \frac{1}{N} \sum_{t=1}^{k} n_t \bar{y}_t \qquad \text{.....(13),}$$

$$\left.\begin{aligned} N v_{110} &= \sum_{t=1}^{k} \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_0)^2 = N s_{x0}^2 \\ N v_{220} &= \sum_{t=1}^{k} \sum_{a=1}^{n_t} (y_{ta} - \bar{y}_0)^2 = N s_{y0}^2 \\ N v_{120} &= \sum_{t=1}^{k} \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_0)(y_{ta} - \bar{y}_0) = N s_{x0} s_{y0} r_0 \end{aligned}\right\} \qquad \text{.....(14),}$$

that is to say $\bar{x}_0$, $\bar{y}_0$, s_{x0}, s_{y0} and r_0 are the means, standard deviations and correlation coefficient obtained on combining the N pairs of observations from the k samples. Further

$$\left.\begin{aligned} N v_{11a} &= \sum_{t=1}^{k} \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_t)^2 = \sum_{t=1}^{k} n_t s_{xt}^2 \\ N v_{22a} &= \sum_{t=1}^{k} \sum_{a=1}^{n_t} (y_{ta} - \bar{y}_t)^2 = \sum_{t=1}^{k} n_t s_{yt}^2 \\ N v_{12a} &= \sum_{t=1}^{k} \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_t)(y_{ta} - \bar{y}_t) = \sum_{t=1}^{k} n_t s_{xt} s_{yt} r_t \end{aligned}\right\} \qquad \text{.....(15),}$$

$$N v_{11m} = \sum_{t=1}^{k} n_t (\bar{x}_t - \bar{x}_0)^2, \quad N v_{22m} = \sum_{t=1}^{k} n_t (\bar{y}_t - \bar{y}_0)^2, \quad N v_{12m} = \sum_{t=1}^{k} n_t (\bar{x}_t - \bar{x}_0)(\bar{y}_t - \bar{y}_0)$$

.....(16).

We shall write for each sample $(t = 1, 2, \ldots k)$

$$\left.\begin{aligned} n_t v_{11t} &= \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_t)^2 = n_t s_{xt}^2, \quad n_t v_{22t} = \sum_{a=1}^{n_t} (y_{ta} - \bar{y}_t)^2 = n_t s_{yt}^2 \\ n_t v_{12t} &= \sum_{a=1}^{n_t} (x_{ta} - \bar{x}_t)(y_{ta} - \bar{y}_t) = n_t s_{xt} s_{yt} r_t \end{aligned}\right\} \qquad \text{.....(17).}$$

Placing these values in (3) and taking the ratio as defined by (8) we find,

$$\lambda_H = \prod_{t=1}^{k} \left[\frac{|v_{ijt}|}{|v_{ij0}|} \right]^{n_t} \qquad \text{.....(18),}$$

where *

$$|v_{ijt}| = \begin{vmatrix} v_{11t} & v_{12t} \\ v_{12t} & v_{22t} \end{vmatrix} = s_{xt}^2 s_{yt}^2 (1 - r_t^2) \qquad \text{.....(19),}$$

$$|v_{ij0}| = \begin{vmatrix} v_{110} & v_{120} \\ v_{120} & v_{220} \end{vmatrix} = s_{x0}^2 s_{y0}^2 (1 - r_0^2) \qquad \text{.....(20),}$$

* For convenience we shall call $|v_{ijt}|$ the generalised variance of the tth sample with elements having $n_t - 1$ degrees of freedom. Similarly, $|v_{ij0}|$, $|v_{ijm}|$ and $|v_{ija}|$ will be generalised variances derived from the combined samples, with elements having $N - 1$, $k - 1$ and $N - k$ degrees of freedom respectively.

(ii) Criterion for H_1. We find that $C(\Omega \text{ max})$ occurs for the same values of the parameters as in the case of H, namely, those given by (9) and (10). $C(\omega \text{ max})$ occurs when (9) is true and when

$$\sigma_x^2 = v_{11a}, \quad \sigma_y^2 = v_{22a}, \quad \sigma_x \sigma_y \rho = v_{12a} \quad \text{.....(21).}$$

Thus it follows that

$$\lambda_{H_1} = \prod_{t=1}^{k} \left[\frac{|v_{ijt}|}{|v_{ija}|} \right]^{\frac{n_t}{2}} \quad \text{.....(22),}$$

where $|v_{ija}|$ is a determinant analogous to those of (19) and (20).

(iii) Criterion for H_2. $C(\Omega \text{ max})$ occurs when (9) and (21) are satisfied and $C(\omega \text{ max})$ when the parameters have the values (11) and (12). Consequently

$$\lambda_{H_2} = \left[\frac{|v_{ija}|}{|v_{ija} + v_{ijm}|} \right]^{\frac{N}{2}} = \left[\frac{|v_{ija}|}{|v_{ij0}|} \right]^{\frac{N}{2}} \quad \text{.....(23)}$$

As in the case of the single-variable problem we observe that

$$\lambda_H = \lambda_{H_1} \times \lambda_{H_2} \quad \text{.....(24).}$$

III. Interpretation of the Criteria.

(5) *The H_2 test.* We note that the structure of each of the λ's given by (18), (22) and (23) differs from that of the corresponding λ of the single-variable problem only in that determinants of the second order matrices of variances and covariances appear in place of the corresponding sums of squares in the single-variable case. In other words, determinants of the second order now take the place of determinants of the first order.

We shall first examine λ_{H_2}; in testing H_2 we have assumed that (6) is true and logically we should first consider the grounds for this assumption, if necessary by testing H_1. The test of H_2 is, however, related to R. A. Fisher's tests in the analysis of variance, and it will be clearer to consider this first.

A λ-criterion must lie between 0 and 1, and if the principle underlying the selection is valid, as it decreases from unity towards zero we should be more and more inclined to reject the hypothesis tested in favour of some one of the admissible alternative hypotheses. How far do our intuitional requirements appear satisfied by λ_{H_2}? The ratio of determinants

$$\lambda_{H_2}^{2/N} = \frac{|v_{ija}|}{|v_{ija} + v_{ijm}|} \quad \text{.....(25)}$$

is of the form of the ratio ψ of Theorem I of the Appendix, if we set $v_{ija} = A_{ij}$, and

$$\sqrt{\frac{n_t}{N}} (\bar{x}_t - \bar{x}_0) = \eta_{1t}, \qquad \sqrt{\frac{n_t}{N}} (\bar{y}_t - \bar{y}_0) = \eta_{2t} \quad \text{.....(26).}$$

It follows that

(*a*) $0 \leqslant \lambda_{H_2} \leqslant 1$; and when $|v_{ija}| > 0$;

(*b*) a necessary and sufficient condition for $\lambda_{H_2} = 1$ is that $\bar{x}_t = \bar{x}_0$, $\bar{y}_t = \bar{y}_0$ $(t = 1, 2, \ldots k)$, that is, that all the sample means be the same;

(*c*) a necessary and sufficient condition that $\lambda_{H_2} = 0$ is that at least one of the differences $\bar{x}_t - \bar{x}_0$, $\bar{y}_t - \bar{y}_0$ $(t = 1, 2, \ldots k)$ be infinite.

It can also be shown by the ordinary methods of differentiation that λ_{H_2} cannot have any other maximum but that of unity (occuring when $\bar{x}_t = \bar{x}_0$ and $\bar{y}_t = \bar{y}_0$) for any given values of the v_{ija}. In fact the maximum is the only stationary point. Therefore, if we keep the intra-sample variation the same, and allow the system to vary from one in which all of the sample means are equal to the other extreme in which at least one sample mean differs very greatly from the mean of the whole, λ_{H_2} will at the same time decrease from 1 to 0.

The case where
$$|v_{ija}| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$
needs special consideration. The determinant is essentially of the form
$$\begin{vmatrix} \Sigma \xi_{ta}^2 & \Sigma \xi_{ta}\eta_{ta} \\ \Sigma \xi_{ta}\eta_{ta} & \Sigma \eta_{ta}^2 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$
where
$$\Sigma = \sum_{t=1}^{k} \sum_{a=1}^{n_t}, \quad \xi_{ta} = \frac{(x_{ta} - \bar{x}_t)}{\sqrt{N}} \text{ and } \eta_{ta} = \frac{(y_{ta} - \bar{y}_t)}{\sqrt{N}};$$
which can vanish only when $\xi_{ta} = c\eta_{ta}$, i.e. when $x_{ta} - \bar{x}_t = c(y_{ta} - \bar{y}_t)$ where c is a constant for all t and a, which may be finite or infinite. In this case the observation points (x_{ta}, y_{ta}) for each sample lie on a straight line, and the lines are all parallel. If $c = 0$ the lines will be horizontal as in Fig. 1 (*a*), if $c = \pm\infty$ they will be vertical as in (*b*), otherwise they will be sloping as in (*c*)*. Hypothesis H_2 is exceedingly improbable (and $\lambda_{H_2} = 0$) unless these lines coincide, which will occur only when
$$x_{ta} - \bar{x}_0 = c(y_{ta} - \bar{y}_0),$$
that is when $\bar{x}_t - \bar{x}_0 = c(\bar{y}_t - \bar{y}_0)$. In this case $\lambda_{H_2} = 0/0$, and we are really reduced to a single-variable problem, and could apply the appropriate H_2 test for that case. It appears therefore that the criterion λ_{H_2} does satisfy our intuitional requirements, at any rate as far as the limiting values 1 and 0 are concerned.

(6) *Alternatives to* λ_{H_2}. In the one-variable problem λ_{H_2} is expressible in terms of η^2, the squared correlation ratio, and hence also in terms of $1 - \eta^2$†. In fact, using the present notation, and considering the x variable only,
$$\frac{v_{11m}}{v_{11a} + v_{11m}} = \eta^2, \qquad \frac{v_{11a}}{v_{11a} + v_{11m}} = 1 - \eta^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29),$$
and consequently it is immaterial which of the two ratios is regarded as the criterion. In the case of two variables the corresponding ratios will be
$$\frac{|v_{ijm}|}{|v_{ija} + v_{ijm}|} = v_2^2, \qquad \frac{|v_{ija}|}{|v_{ija} + v_{ijm}|} = \lambda_{H_2}^{2/N} = L_2^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30),$$
but here U_2 cannot be expressed as a single valued function of L_2. As will be shown below, the sampling distributions of both L_2 and U_2 are very simple functions, and it is natural to ask whether U_2 might be used as an alternative criterion

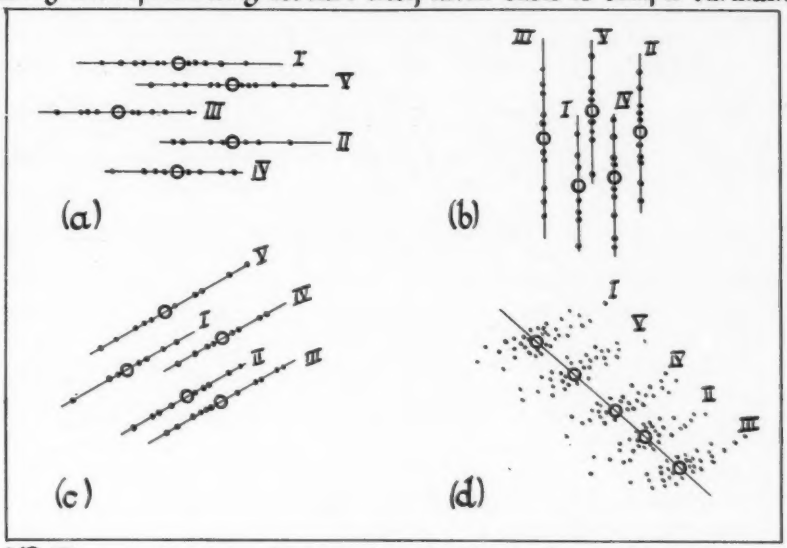
* In the diagram the spots represent observation points (x, y) and the circles represent means of samples.

† Actually $\lambda_{H_2}^{2/N} = 1 - \eta^2$.

for testing hypothesis H_2, decreasing from 1 to 0 as the hypothesis becomes more and more likely. It can be readily seen however that U_2 is not a suitable criterion. Suppose that the v_{ija} are finite and not zero, so that there is variation within the samples; then $U_2 \to 0$ when $|v_{ijm}| \to 0$. This may occur,

(*a*) When $\bar{x}_t \to \bar{x}_0$ and $\bar{y}_t \to \bar{y}_0$ for $t = 1, 2, \ldots k$, i.e. when the means of all samples tend to coincide, and hypothesis H_2 is probable.

Diagram representing certain exceptional cases of sample variation



N.B. The numbers are inserted to indicate lines of points corresponding to different samples.

Fig. 1.

(*b*) But since $|v_{ijm}|$ is of the form (28) it follows that it will also tend to zero when $\bar{x}_t - \bar{x}_0 \to c(\bar{y}_t - \bar{y}_0)$, c being the same constant for all t. This would happen when the sample means tend to lie on a straight line, and when, as suggested in Fig. 1 (*d*), hypothesis H_2 may be quite untenable. Clearly therefore U_2 is not an acceptable criterion.

There are two other forms of alternative criteria which it is of interest to refer to here. On the assumption that the samples have been drawn from identical normal populations, it is possible to obtain from the data two independent estimates of both,

(*a*) $\sigma_x \sigma_y \sqrt{1 - \rho^2}$,

(*b*) $\sigma_x \sigma_y \rho$.

Case (*a*). If we write $\theta = |v_{ija}|$, $\phi = |v_{ijm}|$, then it is known* that

$$df(\theta) = \frac{2^{N-k-2}\,\Delta^{\frac{1}{2}(N-k-1)}}{\Gamma(N-k-1)}\,\theta^{\frac{1}{2}(N-k-3)}\,e^{-2\sqrt{\Delta\theta}}\,d\theta \quad \ldots\ldots\ldots\ldots(31),$$

$$df(\phi) = \frac{2^{k-3}\,\Delta^{\frac{1}{2}(k-2)}}{\Gamma(k-2)}\,\phi^{\frac{1}{2}(k-4)}\,e^{-2\sqrt{\Delta\phi}}\,d\phi \quad \ldots\ldots\ldots\ldots(32),$$

where $\Delta = N^2/[4\sigma_x^2\sigma_y^2(1-\rho^2)]$. Furthermore θ and ϕ are independently distributed, and it may be readily shown from (31) and (32), using the symbol E for "expected" values, that

$$E(\sqrt{\theta}) = \frac{N-k-1}{N}\,\sigma_x\sigma_y\sqrt{1-\rho^2}, \qquad E(\sqrt{\phi}) = \frac{k-2}{N}\,\sigma_x\sigma_y\sqrt{1-\rho^2} \ldots(33).$$

Hence $\frac{N}{N-k-1}\sqrt{|v_{ija}|}$ and $\frac{N}{k-2}\sqrt{|v_{ijm}|}$ may be taken as independent estimates of $\sigma_x\sigma_y\sqrt{1-\rho^2}$ with elements having $N-k$ and $k-1$ degrees of freedom respectively. If we now take the ratio of the two estimates, or

$$\psi = \frac{N-k-1}{k-2}\sqrt{\frac{|v_{ijm}|}{|v_{ija}|}} \quad \ldots\ldots\ldots\ldots(34),$$

it is found from (31) and (32) that

$$df(\psi) = \frac{\Gamma(N-3)}{\Gamma(N-k-1)\,\Gamma(k-2)}(k-2)^{k-2}(N-k-1)^{N-k-1} \times \psi^{k-3}[(k-2)\psi + (N-k-1)]^{(-N-3)}\,d\psi \ldots(35).$$

This is a Pearson Type VI curve; the 5% and 1% sampling limits for ψ could be obtained by taking

$$z = \tfrac{1}{2}\log_e\psi \quad \ldots\ldots\ldots\ldots(36),$$

and entering R. A. Fisher's† tables of z with $2(k-2)$ and $2(N-k-1)$ degrees of freedom.

The criterion ψ, will not, however, be suitable for testing the hypothesis H_2. We may write

$$|v_{ijm}| = s_{xm}^2\,s_{ym}^2(1-r_m^2) \quad \ldots\ldots\ldots\ldots(37),$$

where s_{xm}, s_{ym} and r_m are the standard deviations and coefficient of correlation of the weighted sample means. Then clearly, it would be possible for ψ to be unity and $|v_{ija}|$ fixed and finite, while $r_m \to 1$ and either s_{xm} or $s_{ym} \to \infty$. In such a situation H_2 would be untenable and yet the two independent estimates of $\sigma_x\sigma_y\sqrt{1-\rho^2}$ would be equal.

Case (*b*). It can be easily shown that $\frac{Nv_{12a}}{N-k}$ and $\frac{Nv_{12m}}{k-1}$ are two independent estimates $\sigma_x\sigma_y\rho$, but their ratio, say ψ', would again be unsuitable for testing H_2 for reasons similar to those holding in the case of ψ. It should also be pointed out that the sampling distribution of ψ' is extremely complicated.

* S. S. Wilks: *loc. cit.* p. 477.

† R. A. Fisher: *Statistical Methods for Research Workers*, 4th edition 1932. Edinburgh: Oliver and Boyd.

These illustrations bring out forcibly an important but often neglected consideration. A critical examination of the efficiency of any statistical criterion is necessary before it is applied to testing a hypothesis. The fact that its sampling distribution is known if the hypothesis be true, does not by itself justify its use. In the present case in using U_2, ψ or ψ' we should be in danger of accepting the hypothesis H_2 in certain cases when it is evidently not true. It is only the likelihood criterion, λ_{H_2}, which appears suitable for our purpose.

(7) *The H_1 test.* λ_{H_1} has been defined by (22); clearly $\lambda_{H_1}^{2/N}$ is of the form of the ratio θ discussed in Theorem II of the Appendix, and it satisfies all of the conditions of that theorem. It follows that $\lambda_{H_1}^{2/N}$ and consequently λ_{H_1}

(a) must lie between 0 and 1;

(b) will be unity when and only when $v_{ijt} = v_{iju}$, $(i, j = 1, 2)$ for all values of t and u, that is, when the variances and covariances of x and y are respectively equal in all the k samples;

(c) will be zero when, (i) $x_{t\alpha} - \bar{x}_t = c_t (y_{t\alpha} - \bar{y}_t)$ for at least one value of t, where c_t is a constant for all values of α; this means that the sample points in at least one sample will lie on a straight line, and there is perfect correlation in some but not all samples. However, if there be a c such that $x_{t\alpha} - \bar{x}_t = c\,(y_{t\alpha} - \bar{y}_t)$ for all values of α and t, λ_{H_1} has the indeterminant form 0/0, and the points of each of the samples will lie on a straight line and the lines will all be parallel. In this case the problem is reduced to that of a single variable, and the appropriate H_1 test could be applied; (ii) one of the deviations $x_{t\alpha} - \bar{x}_t$ or $y_{t\alpha} - \bar{y}_t$ or both are infinite for at least one value of α and t (but not all values of t, assuming $|v_{ijt}| > 0$ for all t), subject to the condition that the limiting values of the generalised variances remain finite and not zero as these deviations become infinite. Under these conditions it follows by the argument of the proof of (c) in Theorem I of the Appendix that $|v_{ijo}|$ becomes infinite while the $|v_{ijt}|$ remain finite and different from zero. The situation is a limiting form of that in which the variation is very much greater in some samples than in others.

(8) *The H test.* Since λ_H is the product of λ_{H_1} and λ_{H_2} it must lie between 0 and 1. It can be unity only when both λ_{H_1} and λ_{H_2} are unity, that is to say, when the means, variances and covariances of x and y in all k samples are respectively identical. It will approach zero when λ_{H_1} or λ_{H_2} or both approach zero.

As in the single-variable problem, the three λ ratios appear therefore to satisfy our intuitive requirements as criteria for testing H, H_1 and H_2, for they are quantities which tend to unity as the corresponding hypothesis becomes intuitively more probable (as far as the information contained in the sample is concerned) and tend to zero as it becomes more likely that the hypothesis is false. Whether the tests based on these criteria satisfy the more fundamental conditions laid down by Neyman and Pearson *, we do not yet know. The problem of testing these hypotheses

* *Phil. Trans. Roy. Soc.*, Ser. A, Vol. 231 (1933).

will be completed by determining the sampling distributions of the λ's on the assumption that the corresponding hypotheses are true, for without these we have no means of testing the significance of an observed value of λ. In the following section we shall first give expressions for the moment coefficients, and then by inverting the moment equations show how the frequency distributions may be obtained. The result is simple only in the case of λ_{H_2}; for λ_H and λ_{H_1} numerical values for the probability integrals can be obtained only by some method of approximation.

IV. The Moment Coefficients and Distributions of the Criteria.

(9) *The Moment Coefficients.* In the single-variable problem it was found to be convenient to study the sampling distributions of some fractional power of the λ's, rather than that of the λ's themselves, owing to the extreme skewness of the latter distributions*. The use of $\lambda^{2/N}$ was suggested largely because in this case

$$\lambda_{H_2}^{2/N} = 1 - \eta^2,$$

(where η^2 is the squared correlation ratio) and had a sampling distribution of Type I form. In the present bivariate case we shall find for similar reasons, which will become apparent as we proceed, that some advantage will be gained by using the $\frac{1}{N}$-th power of the λ's. The moments of the λ's have been given by one of the writers† in a recent paper for the case of an n-variate normal system. If we denote by M_{0h}, M_{1h} and M_{2h} the h-th moment coefficients about zero of $\lambda_H^{1/N}$, $\lambda_{H_1}^{1/N}$, and $\lambda_{H_2}^{1/N}$ respectively, when the corresponding hypotheses H, H_1 and H_2 are true, then we have at once from the paper just cited, for the case of two variables (i.e. $n = 2$),

$$M_{2h} = \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-k}{2}\right)\Gamma\left(\frac{N-k-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{N-k+h}{2}\right)\Gamma\left(\frac{N-k-1+h}{2}\right)}{\Gamma\left(\frac{N-1+h}{2}\right)\Gamma\left(\frac{N-2+h}{2}\right)} \quad \text{.....(38),}$$

$$M_{1h} = \frac{\Gamma\left(\frac{N-k}{2}\right)\Gamma\left(\frac{N-k-1}{2}\right)}{\Gamma\left(\frac{N-k+h}{2}\right)\Gamma\left(\frac{N-k-1+h}{2}\right)} \times \prod_{t=1}^{k} \left\{ \left(\frac{N}{n_t}\right)^{\frac{hn_t}{N}} \frac{\Gamma\left(\frac{n_t-1}{2} + \frac{hn_t}{2N}\right)\Gamma\left(\frac{n_t-2}{2} + \frac{hn_t}{2N}\right)}{\Gamma\left(\frac{n_t-1}{2}\right)\Gamma\left(\frac{n_t-2}{2}\right)} \right\} \quad \text{.....(39),}$$

$$M_{0h} = M_{1h} \times M_{2h} \quad \text{.....(40).}$$

* Cf. J. Neyman and E. S. Pearson: *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A (1931), pp. 475—476.

† S. S. Wilks: *Biometrika*, Vol. XXIV (1932), pp. 471– 494. In this paper the generalisations of λ_H, λ_{H_1} and λ_{H_2} were denoted by $\lambda_{H(n)}$, $\lambda_{H'(n)}$ and λ_H respectively.

These expressions may be considerably simplified by making use of the duplication formula of the Gamma function which can be written

$$\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\alpha+1)=\frac{\Gamma(\tfrac{1}{2})\,\Gamma(2\alpha+1)}{2^{2\alpha}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(41).$$

Applying this to (38) and (39) we get

$$M_{2h}=\frac{\Gamma(N-2)\,\Gamma(N-k-1+h)}{\Gamma(N-k-1)\,\Gamma(N-2+h)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

and

$$M_{1h}=\frac{\Gamma(N-k-1)}{\Gamma(N-k-1+h)}\prod_{t=1}^{k}\left[\left(\frac{N}{n_t}\right)^{\frac{nh_t}{N}}\frac{\Gamma\left(n_t-2+\frac{hn_t}{N}\right)}{\Gamma(n_t-2)}\right] \quad \ldots\ldots\ldots\ldots(43).$$

(10) *The Distributions of L_2 and U_2.* To find the distribution of $\lambda_{H_2}^{1/N}=L_2$, we use the relation

$$\frac{\Gamma(N-k-1+h)}{\Gamma(N-2+h)}=\frac{1}{\Gamma(k-1)}\int_0^1 u^{N-k-2+h}(1-u)^{k-2}\,du \quad \ldots\ldots\ldots\ldots(44)$$

in (42). Accordingly, we find that the h-th moment of L_2 is identical with that of u, where the distribution of u is

$$\frac{\Gamma(N-2)}{\Gamma(N-k-1)\,\Gamma(k-1)}\,u^{N-k-2}(1-u)^{k-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(45).$$

Therefore it follows from the uniqueness of the solution of the moment problem for a finite interval* that the distribution of L_2 must be identical with that of u, and is given by

$$df(L_2)=\frac{\Gamma(N-2)}{\Gamma(N-k-1)\,\Gamma(k-1)}\,L_2^{N-k-2}(1-L_2)^{k-2}\,dL_2 \quad \ldots\ldots\ldots\ldots(46).$$

In a similar manner, it follows that the distribution of U_2 is given by

$$df(U_2)=\frac{\Gamma(N-2)}{\Gamma(N-k)\,\Gamma(k-2)}\,U_2^{k-3}(1-U_2)^{N-k-1}\,dU_2 \quad \ldots\ldots\ldots\ldots(47).$$

In both these cases the probability integral is an incomplete B-Function.

(11) *The Distribution of L_1.* Let us consider the sampling distribution of

$$\lambda_{H_1}^{1/N}=L_1,$$

say. The h-th moment of L_1 about zero is given by (43). If we multiply and divide (43) by $\Gamma(N-2k+h)$ and then use the following relations,

$$\frac{\Gamma(N-2k+h)}{\Gamma(N-k-1+h)}=\frac{1}{\Gamma(k-1)}\int_0^1 u^{N-2k-1+h}(1-u)^{k-2}\,du \quad \ldots\ldots\ldots\ldots(48)$$

and

$$\frac{\prod_{t=1}^{k}\Gamma(n_t-2+hp_t)}{\Gamma(N-2k+h)}=\int_0^1\int_0^{l_1}\cdots\int_0^{l_{k-2}}(1-v_1-v_2-\ldots-v_{k-1})^{n_k-3+hp_k}\prod_{t=1}^{k-1}(v_t^{n_t-3+hp_t}\,dv_t) \quad \ldots(49),$$

* See W. Stekloff: *Mémoires de l'Académie Impériale des Sciences de St Pétersbourg*, Vol. XXXIII, No. 9 (1915).

where $l_i = 1 - v_1 - v_2 - \ldots - v_i$ $(i = 1, 2, \ldots k-2)$ and $p_t = \frac{n_t}{N}$ $(t = 1, 2, \ldots k)$, we find that the h-th moment $(h = 0, 1, 2, \ldots)$ of L_1 is identical with that of

$$\phi = \frac{1}{p_1^{p_1} p_2^{p_2} \ldots p_k^{p_k}} u v_1^{p_1} v_2^{p_2} \ldots v_{k-1}^{p_{k-1}} (1 - v_1 - v_2 - \ldots - v_{k-1})^{p_k} \quad \ldots\ldots(50),$$

where u and the v's are distributed according to the function

$$C u^{N-2k-1} (1-u)^{k-2} (1 - v_1 - v_2 - \ldots - v_{k-1})^{n_k - 3} \prod_{t=1}^{k-1} v_t^{n_t - 3} \quad \ldots\ldots(51),$$

where $0 \leqslant u \leqslant 1$, $v_t \geqslant 0$, and $v_1 + v_2 + \ldots + v \leqslant 1$ and C is a constant depending only on k and the n's. Therefore, it follows from the argument used in establishing the uniqueness of (46) that the distribution of L_1 is identical with that of ϕ*. The problem of finding the distribution of ϕ is equivalent to that of solving (50) for the u or one of the v's and substituting in (51) and integrating with respect to all variables except ϕ. This process is extremely complicated, even when the p's are all equal, that is, when $n_1 = n_2 = \ldots = n_k = n$; in this case we can find an expression for the distribution of the L_1 by considering a transformation of M_{1h}. The new form of M_{1h} is found by applying the transformation†

$$\Gamma(mz) = \frac{m^{mz-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(m-1)}} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \ldots \Gamma\left(z + \frac{m-1}{m}\right) \quad \ldots\ldots(52)$$

to $\Gamma(N-k-1)$ and $\Gamma(N-k-1+h)$ in (43), by writing $m = k$ and $z = n - 1 - 1/k$ in the first and $m = k$ and $z = n - 1 - 1/k + h/k$ in the second. Accordingly, we get

$$M_{1h} = C \frac{\Gamma^k\left(n - 2 + \frac{h}{k}\right)}{\Gamma\left(n - \frac{k+1}{k} + \frac{h}{k}\right) \Gamma\left(n - \frac{k}{k} + \frac{h}{k}\right) \ldots \Gamma\left(n - \frac{2}{k} + \frac{h}{k}\right)} \quad \ldots\ldots(53),$$

where

$$C = \frac{\Gamma\left(n - \frac{k+1}{k}\right) \Gamma\left(n - \frac{k}{k}\right) \ldots \Gamma\left(n - \frac{2}{k}\right)}{\Gamma^k(n-2)}.$$

Distribution functions with moments of this type have been considered by one of the authors‡, from which we can write at once as the distribution of L_1

$$df(L_1) = C' L_1^{N-3k} (1 - L_1^k)^{2k - \frac{(k+1)(k+2)-2}{2k} - 1} d(L_1^k) \times \int_0^1 \int_0^1 \ldots \int_0^1 \theta_1^{\frac{k-1}{k} - 1} \theta_2^{\frac{k}{k} - 1} \ldots \theta_{k-1}^{\frac{2k-3}{k} - 1}$$

* In this connection we note that a simple alternative proof of (*a*) in Theorem II of the Appendix can be constructed at once for the case where the p's are rational numbers and the a_{ijt} are product moments, by showing that the maximum of ϕ is unity for variations of u and the v's in the region over which (51) is defined. Indeed, for a given value of u we find that the only stationary point with respect to the v's is the true maximum which occurs where $v_t = p_t$ $(t = 1, 2, \ldots k-1)$. Therefore, ϕ, which is necessarily positive, has a maximum of unity, and since the range of ϕ and L_1 must be the same we have $0 \leqslant L_1^a \leqslant 1$, that is, $0 \leqslant \theta \leqslant 1$.

† See Whittaker and Watson: *Modern Analysis* (4th edition), p. 240.

‡ Wilks, *loc. cit.* pp. 474—475.

$$\times (1-\theta_1)^{2(k-1)-\frac{k(k+1)-2}{2k}-1}(1-\theta_2)^{2(k-2)-\frac{(k-1)k-2}{2k}-1}\dots$$
$$\dots(1-\theta_{k-1})^{2-\frac{2\cdot 3-2}{2k}-1}$$
$$\times[1-\theta_1(1-L_1{}^k)]^{-\frac{k}{k}}[1-\{\theta_1+\theta_2(1-\theta_1)\}(1-L_1{}^k)]^{-\frac{k+1}{k}}\dots$$
$$\times[1-\{\theta_1+\theta_2(1-\theta_1)+\dots+\theta_{k-1}(1-\theta_1)\dots$$
$$\dots(1-\theta_{k-2})\}(1-L_1{}^k)]^{-\frac{2k-2}{k}}\,d\theta_1\dots d\theta_{k-1}\quad\dots\dots(54),$$

where (using formula (52))

$$C'=\frac{C}{\Gamma\left(\frac{k-1}{k}\right)\Gamma\left(\frac{k}{k}\right)\dots\Gamma\left(\frac{2k-2}{k}\right)}=\frac{\Gamma(N-k-1)}{\Gamma^k(n-2)\,\Gamma(k-1)\,k^{N-2k}},$$

a slightly more condensed form of (54) can be obtained by setting

$$\theta_t=1-\phi_t\quad(t=1,2,\dots k-1).$$

Thus
$$df(L_1)=C'L_1{}^{nk-2k}(1-L_1{}^k)^{2k-\frac{(k+1)(k+2)-2}{2k}-1}\,d(L_1{}^k)$$
$$\times\int_0^1\int_0^1\dots\int_0^1\phi_1{}^{2(k-1)-\frac{k(k+1)-2}{2k}-1}\phi_2{}^{2(k-2)-\frac{(k-1)k-2}{2k}-1}\dots\phi_{k-1}{}^{2-\frac{2}{k}-1}$$
$$\times[(1-\phi_1)^{\frac{k-1}{k}-1}(1-\phi_2)^{\frac{k}{k}-1}(1-\phi_3)^{\frac{k+1}{k}-1}\dots(1-\phi_{k-1}{}^{\frac{2k-3}{k}-1}]$$
$$\times[1-(1-\phi_1)(1-L_1{}^k)]^{-1}[1-(1-\phi_1\phi_2)(1-L_1{}^k)]^{-1-\frac{1}{k}}\dots$$
$$\dots[1-(1-\phi_1\phi_2\dots\phi_{k-1})(1-L_1{}^k)]^{-2+\frac{2}{k}}\,d\phi_1 d\phi_2\dots d\phi_{k-1}$$
$$\dots\dots(55).$$

The distribution of L_1 for two samples ($k=2$) turns out to be

$$df(L_1)=\frac{\Gamma(2n-3)}{\Gamma^2(n-2)\,2^{2n-6}}L_1{}^{2n-5}\log\left(\frac{1+\sqrt{1-L_1{}^2}}{L_1}\right)dL_1\quad\dots\dots\dots(56),$$

and the significance of an observed value L_1 can be obtained from the probability integral (57), which results from integrating (56) by parts.

$$P(L_1<l_1)=\int_0^{l_1}df(L_1)$$
$$=\frac{\Gamma(2n-3)}{\Gamma(n-1)\,\Gamma(n-2)\,2^{2n-5}}\left\{(l_1{}^2)^{n-2}\log\left(\frac{1+\sqrt{1-l_1{}^2}}{l_1}\right)+\tfrac{1}{2}\int_0^{l_1{}^2}y^{n-3}(1-y)^{-\frac{1}{2}}\,dy\right\}$$
$$\dots\dots(57).$$

This expression depends on the Incomplete Beta Function. When $k>2$ we have so far been unable to find any simple expression for $df(L_1)$, and some method of approximation seems necessary. Approximate methods are discussed below.

(12) *The Distribution of L.* In a similar manner, if we let $\lambda_H^{1/N}=L$ then the h-th moment of L when $n_1=n_2=\dots=n_k=n$ is

$$M_{0h}=\frac{\Gamma(nk-2)}{\Gamma(nk-2+h)}\,k^h\,\frac{\Gamma^k\left(n-2+\frac{h}{k}\right)}{\Gamma^k(n-2)}\quad\dots\dots\dots\dots\dots(58),$$

and the distribution of L can be expressed as

$$df(L) = \frac{\Gamma(nk-2)}{\Gamma^k(n-2)\,\Gamma(2k-2)\,k^{nk-2k}} L^{(n-3)k}(1-L^k)^{2k+\frac{(k-3)(k-2)-6}{2k}-1} d(L^k)$$

$$\times \int_0^1\int_0^1 \cdots \int_0^1 \phi_1^{2(k-1)+\frac{(k-3)(k-2)-2}{2k}-1} \phi_2^{2(k-2)+\frac{(k-3)(k-2)}{2k}-1} \cdots \phi_{k-1}^{2+\frac{k-3}{k}-1}$$

$$\times (1-\phi_1)^{2-\frac{2}{k}-1}(1-\phi_2)^{2-\frac{1}{k}-1} \cdots (1-\phi_{k-1})^{2+\frac{k-4}{k}-1}[1-(1-\phi_1)(1-L^k)]^{-2+\frac{1}{k}}$$

$$\times[1-(1-\phi_1\phi_2)(1-L^k)]^{-2} \cdots [1-(1-\phi_1\phi_2\cdots\phi_{k-1})(1-L^k)]^{-2-\frac{k-3}{k}}$$

$$d\phi_1 \cdots d\phi_{k-1} \quad \ldots\ldots(59).$$

For the case $k=2$, (59) becomes

$$df(L) = \frac{\Gamma(2n-2)}{\Gamma^2(n-2)2^{2n-6}} L^{2n-5}\left\{\log\left(\frac{1+\sqrt{1-L^2}}{L}\right) - \sqrt{1-L^2}\right\} dL \quad \ldots(60).$$

The probability integral of (60) assumes the form

$$P(L<l) = \int_0^l df(L) = \frac{\Gamma(2n-2)}{\Gamma(n-1)\,\Gamma(n-2)\,2^{2n-5}}\left\{(l^2)^{n-4}\left[\log\left(\frac{1+\sqrt{1-l^2}}{l}\right)\right.\right.$$

$$\left.\left. - (1-l^2)^{\frac{1}{2}}\right] + \tfrac{1}{2}\int_0^{l^2} y^{n-3}(1-y)^{\frac{1}{2}}\,dy\right\} \quad \ldots(61).$$

Again, if $k>2$ some approximate solution appears necessary.

We note from the distributions of L_1 and L that (55) and (59) are actually the distributions of the n-th roots of λ_{H_1} and λ_H respectively.

(13) *Approximate Solution for testing* H_1. When $k>2$ it appears necessary to employ some approximate method to calculate the probability integral of the sampling distribution of λ_{H_1} or of $L_1 = \lambda_{H_1}^{1/N}$. To establish the best and simplest method of procedure fuller investigation is required, but we believe that the relatively simple form of approximation which has been used in the single variable case is also suitable here. This involves the assumption that the sampling distribution of L_1 may be represented by the law

$$f(L_1) = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\,\Gamma(m_2)} L_1^{m_1-1}(1-L_1)^{m_2-1} \quad \ldots\ldots\ldots\ldots\ldots(62),$$

where m_1 and m_2 are determined so that the first and second moment coefficients of $f(L_1)$ about $L_1=0$ have the values given by (43) for $h=1$ and 2 respectively. In other words we represent the distribution of L_1 by a Type I curve having the correct terminals and first two moment coefficients. In many practical applications it is possible to plan for the number of individuals in each sample to be the same, i.e. for $n_t = n\,(t=1, 2, \ldots k)$. This is the situation considered in the illustrations which follow. The equations for determining m_1 and m_2 then become

$$m_1 = \frac{M_{11}(M_{11}-M_{12})}{M_{12}-M_{11}^2}, \qquad m_2 = \frac{(1-M_{11})(M_{11}-M_{12})}{M_{12}-M_{11}^2} \quad \ldots\ldots\ldots(63),$$

from which, since $N = nk$, we obtain from (43)

$$M_{11} = \frac{k}{N-k-1}\left\{\frac{\Gamma\left(n-2+\frac{1}{k}\right)}{\Gamma(n-2)}\right\}^k \quad \text{.....(64),}$$

$$M_{12} = \frac{k^2}{(N-k-1)(N-k)}\left\{\frac{\Gamma\left(n-2+\frac{2}{k}\right)}{\Gamma(n-2)}\right\}^k \quad \text{.....(65).}$$

For the probability integral of (62), we have

$$P(L_1 \leqslant l_1) = I_{l_1}(m_1, m_2) \quad \text{.....(66),}$$

which is the Incomplete Beta Function. This may be obtained from the *Tables of the Incomplete Beta Function** if m_1 and m_2 are $\leqslant 50$ or by means of R. A. Fisher's z-transformation as has been suggested elsewhere†. If m_1 and m_2 are both large or nearly equal, (62) will approach the normal form and the ratio

$$\frac{L_1 - \text{Mean } L_1}{\sigma_{L_1}} = \frac{L_1 - M_{11}}{\sqrt{M_{12} - M_{11}^2}} \quad \text{.....(67)}$$

can be used as an index of significance to be interpreted on the normal probability scale.

The probability integral of $L = \lambda_H^{1/N}$ could be obtained by a similar approximation, but in general it is likely that the hypotheses H_1 and H_2 will be tested separately.

V. PRACTICAL ILLUSTRATIONS.

(14) *Example* 1. *Relation between Tensile Strength and Hardness in Aluminium Die-Castings.*

This example has been referred to above in Section (3). We shall proceed first to test H_1, that is, the hypothesis that there is no significant difference between the samples as regards variation and covariation in strength and hardness. A summary of the necessary calculations is shown in Table II; we have $N = 60$, $k = 5$, $n = 12$. The unit for x (strength) is 1000 lb. per square inch, and for y it is Rockwell's E.

From (22) we have

$$L_1 = \lambda_{H_1}^{1/N} = \sqrt{\frac{\prod_{t=1}^{k}\left(|v_{ijt}|\right)^{\frac{1}{k}}}{|v_{ija}|}} \quad \text{..... (68),}$$

and as indicated in the table it is found that $L_1 = \cdot 9065$. From (64) and (65) it is found that if H_1 were true‡, then Mean $L_1 = M_{11} = \cdot 889274$, $M_{12} = \cdot 792592$,

* To be issued shortly as a *Biometrika* publication.

† *Biometrika*, Vol. XXIV. p. 415.

‡ Brownlee's Seven-Figure Tables of the Logarithm of the Gamma Function were used, *Tracts for Computers*, No. IX.

TABLE II.

Strength (x) and Hardness (y) in Aluminium Die-Castings. Test of H_1 (bivariate).

t (Sample No.)	Sums of Squares		Sums of Products	Generalised Variances	
	$nv_{11t} = \sum_{a=1}^{n} (x_{ta} - \bar{x}_t)^2$	$nv_{22t} = \sum_{a=1}^{n} (y_{ta} - \bar{y})^2$	$nv_{12t} = \sum_{a=1}^{n} (x_{ta} - \bar{x}_t)(y_{ta} - \bar{y})$	$\lvert v_{ijt} \rvert = v_{11t}v_{22t} - v_{12t}^2$	$\log \lvert v_{ijt} \rvert$
1	78·948	1247·18	214·18	365·204	2·56254
2	223·695	2519·31	657·62	910·401	2·95923
3	57·448	1241·78	190·63	243·029	2·38566
4	187·618	1473·44	375·91	938·451	2·97241
5	88·456	1171·73	259·18	253·281	2·40360
Totals	636·165 $= Nv_{11a}$	7653·44 $= Nv_{22a}$	1697·52 $= Nv_{12a}$		13·28344 $= \sum_{t=1}^{5} \log(\lvert v_{ijt} \rvert)$

$$\lvert v_{ija} \rvert = v_{11a}v_{22a} - v_{12a}^2 = 552{\cdot}018,$$

$$\log L_1 = \tfrac{1}{2}\left\{\frac{1}{k}\log \prod_{t=1}^{k} (\lvert v_{ijt} \rvert) - \log(\lvert v_{ija} \rvert)\right\} \quad \text{(from definition (68))}$$

$$= \bar{1}{\cdot}957367,$$

$$L_1 = {\cdot}9065.$$

Estimate of correlation within samples $\Big\}$ $r_a = \dfrac{Nv_{12a}}{\sqrt{Nv_{11a} \times Nv_{22a}}} = +{\cdot}7693.$

$\sigma_{L_1} = {\cdot}04223$. The observed value of L_1 is therefore nearer to unity than the mean value expected in repeated samples, and the ratio (67) is only + 0·41. Therefore there is clearly no reason for rejecting H_1. If, however, we were to proceed in more detail we should find from (63) that $m_1 = 48{\cdot}210$, and $m_2 = 6{\cdot}003$, and by interpolating in the Tables of the Incomplete Beta Function that $P(L_1 < {\cdot}9065) = {\cdot}621$.

We may now proceed to test H_2, the hypothesis that neither mean strength nor mean hardness differs significantly from sample to sample. Table III contains a summary of the calculations in the form of an analysis of variance table. It is seen that $L_2 = {\cdot}6896$, while if H_2 were true, the probability law for L_2 is obtained from (46) as

$$P(L_2) = \frac{\Gamma(58)}{\Gamma(54)\,\Gamma(4)} L_2^{53}(1 - L_2)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (69).$$

The mean of (69) is ·9310 and the standard error* is ·0330, so that the observed value differs from the mean by more than seven times the standard error. By actually integrating (69) it is found that $P(L_2 < {\cdot}6896) = {\cdot}0000019$. H_2 must clearly be rejected. To discover whether this is due to significant differences in mean strength or in mean hardness or in both, we must consider the two single-variable problems

* For (46): Mean $L_2 = \dfrac{N-k-1}{N-2}$, $\sigma_{L_2} = \dfrac{1}{N-2}\sqrt{\dfrac{(N-k-1)(k-1)}{N-1}}$.

TABLE III.

Strength (x) and Hardness (y) in Aluminium Die-Castings. Tests of H_2.

	Degrees of Freedom	Sums of Squares (x)	Sums of Squares (y)	Sums of Products (x, y)	Generalised Variances
Between Samples	$k-1=4$	$Nv_{11m}=306{\cdot}089$	$Nv_{22m}=662{\cdot}77$	$Nv_{12m}=214{\cdot}86$	$\lvert v_{ijm}\rvert=43{\cdot}528$
Within Samples	$N-k=55$	$Nv_{11a}=636{\cdot}165$	$Nv_{22a}=7653{\cdot}42$	$Nv_{12a}=1697{\cdot}52$	$\lvert v_{ija}\rvert=552{\cdot}018$
Totals	$N-1=59$	$Nv_{11o}=942{\cdot}254$	$Nv_{22o}=8316{\cdot}19$	$Nv_{12o}=1912{\cdot}38$	$\lvert v_{ijo}\rvert=1160{\cdot}77$

$$L_2=\lambda_2^{1/N}=\sqrt{\lvert v_{ija}\rvert/\lvert v_{ijo}\rvert}={\cdot}6896,$$

$$\eta_{xt}^2=v_{11m}/v_{11o}={\cdot}3248, \qquad \eta_{yt}^2=v_{22m}/v_{22o}={\cdot}0797.$$

	Estimates of Variance	$\log_{10}$(est.)	Estimates of Variance	$\log_{10}$(est.)
Between Samples	$\frac{Nv_{11m}}{k-1}=76{\cdot}522$	1·883786	$\frac{Nv_{22m}}{k-1}=165{\cdot}69$	2·219296
Within Samples	$\frac{Nv_{11a}}{N-k}=11{\cdot}566$	1·063183	$\frac{Nv_{22a}}{N-k}=139{\cdot}15$	2·143483
Difference		·820603	Difference	·075813
$z=1{\cdot}15129\times$ Difference		·9448	z	·0873

separately. Tests may be applied to the squared correlation ratios η_{xt}^2 and η_{yt}^2, or R. A. Fisher's z-transformation can be used. The necessary calculations are shown in Table III.

Using Woo's tables*, it is found that η_{xt}^2 is clearly significant while η_{yt}^2 is not. Alternatively, referring to Fisher's z-tables† with $n_1=k-1=4$, $n_2=N-k=55$, it is seen that the 5 % point lies at about ·47 and the 1 % at about ·65, showing, as before, that mean strength differs significantly from sample to sample but not mean hardness.

The limited amount of test records available would therefore suggest the following tentative conclusions:

(a) Within the samples the relationship between the two qualities is stable, and represented by $\sigma_x=3{\cdot}401\times10^3$ lb. per sq. in., $\sigma_y=11{\cdot}80$ in Rockwell's E,

* *Tables for Statisticians and Biometricians*, Part II, Table IV.

† *Statistical Methods for Research Workers*, Table VI.

$r_{xy} = +\cdot769$. (The first two values are the square roots of $Nv_{11a}/(N-k)$ and $Nv_{22a}/(N-k)$ respectively, and the last is the value of v_a given in Table II.)

(*b*) While the variation in mean strength from sample to sample is imperfectly controlled, the variation in hardness appears no more than might be expected through chance.

From the practical point of view this is not an altogether satisfactory result and further investigation into the anomaly (*b*) would be necessary before hardness could be used with confidence as an index of strength.

(15) *Example* 2. *Relation between Length and Breadth of Human Skulls.*

The data consist of standard measurements of length and breadth of skull in millimetres obtained for 20 adult males from each of 30 different races or groups*, i.e. $N = 600$, $n = 20$, and $k = 30$. That there would be considerable inter-racial variation for mean length and breadth was obvious, but it seemed to be of interest to examine the hypothesis H_1, that is to say, to test the extent of inter-racial uniformity in the relationship of length to breadth. These characters appear sufficiently nearly normally distributed within a race for the normal-theory tests to be applicable. Length will be denoted by x and breadth by y; a summary of the calculations is shown in Table IV. We find from these that

$$|v_{ija}| = 656{\cdot}369 \qquad \frac{1}{k}\sum_{t=1}^{k} \log\{|v_{ija}|\} = 2{\cdot}644429$$

$$\log|v_{ija}| \quad = 2{\cdot}817148$$

$$\text{Difference} \quad \bar{1}{\cdot}827281$$

$$\log L_1 = \tfrac{1}{2} \times \text{difference} = \bar{1}{\cdot}913640,\ L_1 = {\cdot}8197.$$

From (64) and (65) we obtain Mean $L_1 = M_{11} = {\cdot}923678$, $M_{12} = {\cdot}853317$, $\sigma_{L_1} = {\cdot}0117$. The observed L_1 is below the expected mean value, and the ratio (67) is $-8{\cdot}9$. This is so clearly significant that, without further refinement in calculation, we can say that H_1 is untenable. We must now examine whether this lack of uniformity is present both in the group standard deviations s_{xt} and s_{yt}, and in the correlations r_{xyt}.

For the first problem Neyman and Pearson's single variate test for H_1 may be applied†. This involves the calculation of the sums of the logarithms of the quantities nv_{11t} and of v_{22t} given in Table IV, since in this case

$$L_1 = \lambda_{H_1}^{2/N} = \prod_{t=1}^{k} (v_{iit})^{1/k}/(v_{iia}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70),$$

where $i = 1$ for length and $i = 2$ for breadth. The calculations are shown in Table V, v_{11a} and v_{22a} being obtained from Table IV. It is found that were H_1 true, then:

$$\text{Mean } L_1 = {\cdot}9496, \quad \sigma_{L_1} = {\cdot}0129.$$

* We are indebted to Dr G. M. Morant for providing us with the necessary sources of information.

† For an illustration of the use of this test, see *Biometrika*, Vol. XXIV. p. 415.

TABLE IV.

Length (x) and Breadth (y) of Skulls. Tests of H_1; data for separate samples.

t	Race	nv_{11t}	nv_{22t}	nv_{12t}	$\lvert v_{0t}\rvert$	$\bar{x}_t$	$\bar{y}_t$	s_{xt}	s_{yt}	r_t
1	Aëtas	612·00	300·14	41·50	454·91	172·00	143·92	5·532	3·874	+·097
2	Aleuts	218·20	391·20	57·80	205·05	183·30	149·20	3·303	4·423	+·198
3	Andamanese	216·45	311·25	149·50	112·55	167·95	136·25	3·290	3·945	+·576
4	Anglo-Saxons	975·44	777·45	−12·88	1895·47	191·62	142·45	6·984	6·235	−·015
5	Armenians	551·20	320·20	72·80	427·99	176·80	145·70	5·250	4·001	+·173
6	Australians. (South Australia)	1114·20	280·95	427·30	326·12	195·70	127·05	7·464	3·748	+·764
7	Australians. (Victoria)	1131·20	472·95	−27·20	1335·65	182·80	132·45	7·521	4·863	−·037
8	Copts	722·20	558·55	423·90	559·23	174·30	135·85	6·009	5·285	+·667
9	Dayake. (Borneo)	431·50	881·24	8·50	950·46	174·50	138·97	4·645	6·638	+·014
10	Easter Islanders	834·74	331·64	−59·16	683·33	191·72	133·42	6·460	4·072	−·112
11	Egyptians, 1st Dynasty. (Abydos) ...	518·71	319·07	89·03	393·95	185·62	136·50	5·093	3·994	+·219
12	Egyptians, Modern. (Cairo)	1115·45	382·30	−99·28	1041·45	186·86	133·90	7·468	4·372	−·152
13	Egyptians, Predynastic. (Badari) ...	542·05	356·64	140·23	434·13	184·65	131·42	5·206	4·223	+·319
14	English, Mediaeval. (Hythe)	714·95	748·14	227·08	1208·29	179·45	148·07	5·979	6·116	+·310
15	English, Roman? (Spitalfields)	1021·80	550·75	14·25	1406·38	181·10	143·75	7·148	5·248	+·019
16	English, 17th century. (Farringdon St)	697·80	312·24	207·80	436·75	192·10	145·97	5·907	3·951	+·445
17	Eskimos. (Greenland)	325·75	278·95	123·75	188·88	192·75	133·55	4·036	3·735	+·410
18	Gaunche. (Canary Islands)	327·75	158·55	215·75	13·54	189·25	141·65	4·048	2·816	+·946
19	Hindus. (Bengal)	650·55	700·55	11·95	1139·00	176·85	131·65	5·703	5·918	+·018
20	Indonesians. (Ceram)	887·75	1033·80	153·50	2235·48	177·25	134·90	6·662	7·190	+·160
21	Javanese	1018·14	392·14	112·61	966·43	171·57	141·42	7·135	4·428	+·178
22	Mongols. (Urga)	488·95	212·20	245·70	108·47	189·05	145·30	4·944	3·257	+·763
23	Moriori. (Chatham Islands)	232·45	488·24	33·97	280·84	186·05	142·27	3·409	4·941	+·101
24	Negroes, Teita. (Kenya Colony) ...	562·74	368·50	204·50	413·87	187·72	130·00	5·304	4·292	+·449
25	Papuans. (New Guinea)	404·80	448·95	104·60	426·98	188·40	132·05	4·499	4·738	+·245
26	Tagals. (Philippine Islands)	454·64	303·20	133·70	299·93	178·92	138·80	4·768	3·894	+·360
27	Tasmanians	698·00	237·75	241·00	269·67	190·00	139·25	5·908	3·448	+·592
28	Turks	381·80	393·20	−199·60	275·71	168·10	141·80	4·369	4·434	−·515
29	Swiss. (Münster)	405·75	587·20	11·00	595·34	177·75	151·20	4·504	5·418	+·023
30	Venezuelians	480·00	239·20	86·00	268·55	178·00	140·80	4·899	3·458	+·254
	Totals	18736·96 $= Nv_{11a}$	13137·14 $= Nv_{22a}$	+3139·60 $= Nv_{12a}$						

Consequently we have

	Length (x)	Breadth (y)
Observed L_1	·9000	·9074.
$(L_1 - \text{Mean } L_1)/\sigma_{L_1}$	−3·84	−3·27

TABLE V.

Length and Breadth of Skulls. Tests of H_1 (Single-variate).

	Length (x) $i=1$	Breadth (y) $i=2$
$\frac{1}{30}\sum_{t=1}^{30}\log(20v_{iit})$	2·749829	2·599191
$\log 20$	1·301030	1·301030
$\frac{1}{30}\sum_{t=1}^{30}\log(v_{iit})$	1·448799	1·298161
$\log(v_{iia})$	1·494548	1·340349
$\log L_1$	$\bar{1}$·954251	$\bar{1}$·957812
L_1	·9000	·9074

The divergence shown by the ratios is significant, and it does not seem necessary to enter here into the approximate calculation of the probabilities $P(L_1 < \cdot 9000)$ and $P(L_1 < \cdot 9074)$, (which are both under ·01), since examples have been discussed elsewhere and it is hoped to publish shortly convenient tables for use with the test.

We must now examine the variation among the 30 correlation coefficients r_{xyt} $(t = 1, 2, \ldots 30)$; the best method of procedure is probably as follows:

If x and y are normally correlated it is known that in repeated samples of n that

$$z' = \tfrac{1}{2}\{\log_e(1+r) - \log_e(1-r)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (71)$$

is approximately normally distributed with a standard error of $1/\sqrt{n-3}$*. Consequently, we may test whether k independent values of r differ only through chance fluctuations from some unknown population value of ρ, by calculating

$$\chi^2 = \sum_{t=1}^{k}\{(n_t - 3)(z_t' - \bar{z}')^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72),$$

where $\bar{z}' = \sum_{t=1}^{k}(z_t'/k)$, and entering the (χ^2, P) tables with $k-1$ degrees of freedom (i.e. $n' = k$ in the notation of Elderton's Table). In the present instance it is found that $\chi^2 = 96{\cdot}01$, while $n' = 30$, which is evidently significant. χ, in fact, deviates from the expected value by about 6·3 times the standard error†.

We have found therefore that the covariation in length and breadth of skull within a race cannot be considered as uniform from race to race; further, that while

* R. A. Fisher: *Metron*, Vol. I. No. IV. p. 13. An illustration of using this test with k values of r has been given by L. H. C. Tippett: *The Methods of Statistics* (1931), p. 143.

† This result is obtained by using the rule that when $f = n' - 1$ is large, $\sqrt{2\chi^2} - \sqrt{2f-1}$ is approximately normally distributed about zero with unit standard error.

the standard deviations certainly differ significantly, the lack of uniformity is due in much greater degree to the instability of the correlation coefficient. Having regard to the great variety in the data, these results were to be expected since a "race" is a loosely defined term, and the coefficient of correlation between two measures of size within a group will depend upon the homogeneity of that group. The more similar the skulls are in shape the higher is r likely to be. In considering the possible value of the criterion L_1 in anthropometric work, it should be remembered that, although the present paper is concerned only with the case of two correlated variables, the general theory developed by one of us* is applicable in the case of any number of variables.

Although the difference in mean length and breadth for the different races is so obvious that a statistical test is hardly required, it may be useful to summarise what would be the formal method of approach:

(1) Considering the two variables together it is found that $H_1(x \text{ and } y)$ is quite untenable; therefore we should not proceed to test H_2 (x and y).

(2) $H_1(x)$ and $H_1(y)$ are also improbable, but the differences in the standard deviations s_x and s_y are hardly sufficient to invalidate the tests $H_2(x)$ and $H_2(y)$.

(3) From two tables for analysis of variance similar to those contained in Table III, it is found that $\eta_{xt}^2 = \cdot 6489$ and $\eta_{yt}^2 = \cdot 6294$. If H_2 were true for the case of a single variable, then Mean $\eta^2 = \cdot 0484$, and $\sigma_{\eta^2} = \cdot 0124$. Clearly, therefore, $H_2(x)$ and $H_2(y)$ are untenable, that is, the 30 samples of 20 provide convincing evidence that the racial mean characters differ significantly.

VI. CONCLUSION.

Certain general methods of analysis of multivariate data have been developed by one of us elsewhere. In the present paper the special case of two correlated variables has been taken, in order to illustrate (*a*) the process of reasoning underlying the methods, (*b*) the practical application of the resulting tests, (*c*) their relation to other tests in use. The following points may be emphasised:

(1) It is necessary to recognise that in many problems, hypotheses of the H_1 type need to be tested, as well as those of the H_2 type. The technique of Analysis of Variance does not appear suited to deal with the former when more than two samples are concerned.

(2) In the multivariate problem it would be possible to deal with the variation of each character and the correlation of each pair of characters separately, but the application in the first instance of a single comprehensive test has several advantages. If, for example, on the evidence available H can be accepted, it is unnecessary to proceed to test H_1 and H_2. Similarly, if H_1 (using p variates) can be accepted there should be no need to proceed to the p single-variate H_1 tests and the $\frac{1}{2}p(p-1)$

* Wilks, *loc. cit.*

correlation tests. The same situation arises in dealing with H_2. Even when the comprehensive test is not satisfied, and it is necessary to apply the separate tests in order to locate the source of disturbance, relatively little labour will have been wasted in applying the comprehensive test first. For example, in the case of two variables which has been illustrated, the calculation of $\log|v_{ijt}|$ needed to test H_1 (two variables) involves little extra work when once nv_{11t}, nv_{22t} and nv_{12t} have been computed. But the latter quantities are in any case required if the sample variances and correlations are to be considered separately.

(3) The methods suggested for calculating the significance of a given value of the λ or L criteria are admittedly not in final form. For convenience in practical working, tables to be entered with n and k are needed, which would show certain levels of significance of these criteria. The possibility of forming such tables is under consideration.

(4) It has been assumed throughout that the variables are normally distributed. Some investigations on the stringency of this assumption are in progress.

VII. Appendix.

To assist in the interpretation of λ_H, λ_{H_1} and λ_{H_2} as criteria for testing the corresponding hypotheses H, H_1 and H_2 we shall find it convenient to prove the following theorems.

Theorem I. *Let η_{ik} $(i=1, 2;\ k=1, 2, \ldots m)$ be any set of real numbers, and let the matrix*

$$\begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix}$$

be real and positive definite with $A_{12}=A_{21}$ and

$$\psi=\frac{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11}+\Sigma\eta_{1k}^2 & A_{12}+\Sigma\eta_{1k}\eta_{2k} \\ A_{21}+\Sigma\eta_{1k}\eta_{2k} & A_{22}+\Sigma\eta_{2k}^2 \end{vmatrix}},$$

where the moment products of η's are summed for k from 1 to m; then

(*a*) $0\leqslant\psi\leqslant 1$,

(*b*) *a necessary and sufficient condition that $\psi=1$, when*

$$|A_{ij}|>0, \text{ is that } \eta_{ik}=0 \qquad (i=1, 2;\ k=1, 2, \ldots m),$$

(*c*) *a necessary and sufficient condition that $\psi=0$ when $|A_{ij}|>0$, is that at least one of the η's be infinite.*

Proof: Let the determinants in the numerator and denominator of ψ be called A and B respectively. Then (*a*) can be shown at once by induction, for suppose $B\geqslant A$ for $k=t$, then for $k=t+1$, B can be written as B_t+q_{t+1}, where B_t is the value of B for $k=t$ and q_{t+1} is a positive definite quadratic form in $\eta_{1,t+1}$ and $\eta_{2,t+1}$. Therefore, setting $t=0, 1, 2, \ldots m$, we get

$$B\geqslant\ldots\geqslant B_{t+1}\geqslant B_t\geqslant\ldots\geqslant B_0\equiv A>0,$$

which is equivalent to the proof of (*a*).

The sufficient conditions in (*b*) and (*c*) are obvious. To prove the necessary condition of (*b*) we observe that for $B_{t+1}=B_t$ it is necessary that $\eta_{1,t+1}=\eta_{2,t+1}=0$ since q_{t+1} is positive definite in these two variables. Setting $t=0, 1, 2, \ldots m$, we see that a necessary condition for

$$B=\ldots=B_{t+1}=B_t=\ldots=B_0\equiv A,$$

or $\psi=1$ is that $\eta_{it}=0$ ($i=1, 2$; $k=1, 2, \ldots m$).

To prove the necessary condition in (*c*), we note that at least one member of the non-decreasing set $B \ldots B_{t+1}, B_t \ldots B_1$ must become infinite. Let B_{i+1} be the first B which is infinite, then clearly, $B, \ldots B_{i+3}, B_{i+2}$ will also be infinite. But $B_{i+1}=B_i+q_{i+1}$, where B_i is finite. Therefore q_{i+1} must be positively infinite which can occur only when at least one of the numbers $\eta_{1,i+1}$ and $\eta_{2,i+1}$ is numerically infinite.

The proof of this theorem for the case when A and B are determinants of the n-th order can be carried out in essentially the same way as the one just given for two variables.

THEOREM II. *Let a_{ijt} ($i, j=1, 2$; $t=1, 2, \ldots k$) be any set of real numbers in which $a_{ijt}=a_{jit}$ for $i\neq j$, and such that the matrix $||a_{ijt}||$ is positive definite, and let $A=|A_{ij}|$, where*

$$A_{ij}=\sum_{t=1}^{k} p_t a_{ijt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and the p's are positive such that $\Sigma p_t=1$. Then, if

$$\theta=\frac{\prod_{t=1}^{k}|a_{ijt}|^{p_t}}{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

we have

(*a*) $0\leqslant\theta\leqslant 1$,

(*b*) *a necessary and sufficient condition that $\theta=1$ when $|a_{ijt}|>0$ ($t=1, 2, \ldots k$) is that $a_{ijt}=a_{ijt'}$ ($i, j=1, 2$; $t, t'=1, 2, \ldots k$); that is, that the matrices $||a_{ijt}||$ be identical.*

PROOF: For the one-variable case, that is, when $i=j=1$, the theorem simply states that the weighted geometric mean of a set of positive numbers cannot exceed the weighted arithmetic mean of the set, and that the means can be equal only when all of the numbers are equal. The proof for this case can be found in a number of advanced algebra text-books and will be assumed.

The sufficient condition in (*b*) is obvious. Thus, let us consider the necessary condition. For convenience let

$$\phi=A-G \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

where $$G=\prod_{t=1}^{k} d_t^{p_t} \quad \text{and} \quad d_t=|a_{ijt}|.$$

Then the theorem reduces to the problem of showing that $\phi\geqslant 0$, where the equality will hold only when the matrices $||a_{ijt}||$ are identical. Consider the minimum of ϕ for variations of the a's. If it exists, it will be given by the equations ($t=1, 2, \ldots k$),

$$\frac{\partial\phi}{\partial a_{11t}}=p_t A_{22}-p_t\frac{a_{22t}}{d_t}G=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

$$\frac{\partial\phi}{\partial a_{22t}}=p_t A_{11}-p_t\frac{a_{11t}}{d_t}G=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

$$\frac{\partial\phi}{\partial a_{12t}}=-2p_t A_{12}+2p_t\frac{a_{12t}}{d_t}G=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

By a straightforward combination of the first equation for all values of t,

$$A_{22}=\prod_{t=1}^{k} a_{22t}^{p_t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)}.$$

Similarly, with respect to (v),

$$A_{11}=\prod_{t=1}^{k} a_{11t}^{p_t} \quad \text{.....(viii).}$$

From (vi)

$$a_{12t}d_{t'}=a_{12t'}d_t \qquad (t,\ t'=1,\ 2,\ \ldots\ k) \quad \text{.....(ix).}$$

But from the case of a single variable (vii) and (viii) can hold only when

$$a_{11t}=a_{11t'}, \quad a_{22t}=a_{22t'} \qquad (t,\ t'=1,\ 2,\ \ldots\ k) \quad \text{.....(x).}$$

Call their common values a_{11} and a_{22} respectively.

Placing these values in (ix) which must hold for all values of a_{11} and a_{22}, we get at once that

$$a_{12t}=a_{12t'} \qquad (t,\ t'=1,\ 2,\ \ldots\ k) \quad \text{.....(xi).}$$

Let the common value be a_{12}. Therefore (iv), (v) and (vi) are satisfied only when the matrices $||a_{ijt}||$ are identical. The matrix of second order derivatives of ϕ with respect to the a_{ijt} can be shown to be positive definite when (iv), (v) and (vi) are satisfied, provided $||a_{ij}||$ is positive definite. Thus, ϕ has a true minimum, and since the minimum is zero, and G is positive (a) follows at once. The generalisation of this theorem to the case where $i, j=1,\ 2,\ \ldots\ n$ is straightforward.

ON A METHOD OF DETERMINING WHETHER A SAMPLE OF SIZE n SUPPOSED TO HAVE BEEN DRAWN FROM A PARENT POPULATION HAVING A KNOWN PROBABILITY INTEGRAL HAS PROBABLY BEEN DRAWN AT RANDOM.

By KARL PEARSON.

Probability integrals have now been tabled for the following curves, all of which occur frequently in statistics, either as accurate or approximate distributions of statistical quantities:

(i) $y = y_0 e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$, the "normal" curve (probability integral provided by Sheppard's Tables*).

(ii) $y = y_0 x^{p-1}(1-x)^{q-1}$ (Pearson's Type I, *Tables of the Incomplete* B-*Function*†).

(iii) $y = y_0 \left(1 - \frac{x^2}{a^2}\right)^m$ (Pearson's Type II, ditto).

(iv) $y = y_0 \Big/ \left(1 + \frac{x^2}{a^2}\right)^m$ (Pearson's Type VII, "Student's" z-Curve, *Tables of the Incomplete* B-*Function*).

(v) $y = y_0 e^{-\frac{\gamma x}{a}} \left(1 + \frac{x}{a}\right)^p$ (Pearson's Type III, *Tables of the Incomplete* Γ-*Function*).

(vi) $y = y_0 (x-a)^{m_1}/x^{m_2}$ (Pearson's Type VI, by transformation to the Incomplete B-function).

(vii) $y = y_0 e^{-\gamma/x} x^{-p}$ (Pearson's Type V, by transformation to the Incomplete Γ-function).

(viii) $y = y_0 e^{-x/\sigma}$ (Pearson's Type X, Newman and Glaisher's tables‡ of the Exponential Function).

In these and a number of other cases the probability integral can be found or easily computed. The probability integral of the $T_\nu(x)$ Bessel function curve has also been provided by Miss F. N. David§. The sole outstanding case among the Pearson curves is Type IV, $y = y_0 e^{-\nu \tan^{-1}\frac{x}{a}} \Big/ \left(1 + \frac{x^2}{a^2}\right)^m$, where a table of

$$\int_{-\frac{\pi}{2}}^{\theta} e^{-\nu\theta} \cos^m \theta \, d\theta$$

* *Biometrika*, Vol. II. pp. 174—190, or *Tables for Statisticians*, Part I. pp. 1—10.

† Published by *Biometrika*, 1933.

‡ *Camb. Phil. Soc. Trans.* Vol. XIII. Part iii. pp. 145—272.

§ *Biometrika*, Vol. XXIV. pp. 344—346.

for values of ν and n is required. By aid of these probability integral tables any single value of a variate x_s supposed to belong to one of these curves can have its probability integral p_s determined, which measures the frequency of values arising as great as or greater than x_s. Let us suppose a sample of size n containing the variates $x_1, x_2, x_3, \ldots x_s, \ldots x_n$ to be drawn from a distribution $y = \phi(x)$ and let $p_1, p_2, p_3, \ldots p_s, \ldots p_n$ be their respective probability integrals. What will be the distribution of these probability integrals?

Now p has a definite value for each value of x and by definition of the probability integral

$$p = \int_a^x \phi(x)\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $x = a$ is the start of $\phi(x)$ and the whole area $\int_a^b \phi(x)\, dx = 1$.

Now the frequency with which p_s occurs may be represented by $F(p)\, dp$, but this is the same as the frequency of $x = \phi(x)\, dx$, or

$$F(p)\, dp = \phi(x)\, dx,$$

but by (i)
$$dp = \phi(x)\, dx.$$

Accordingly $F(p) = 1$, or *the distribution for the probability integrals of any frequency curve is a rectangle on the base* $p = 0$ *to* $p = 1$*.

We note that this is not a discrete but a perfectly continuous distribution of frequency. Since all the x's are supposed to be obtained by random sampling, all the p's will likewise be a random sampling distribution, and $P_{1\ldots n}$ the probability of that random sample occurring or one with the individual variates having a greater probability, will be given by

$$P_{1\ldots n} = p_1 p_2 \ldots p_s \ldots p_n = \prod_1^n (p_s).$$

We now add: Is this an improbable sample? In other words we enquire: What is the probability of a sample with as great as or a greater value than $P_{1\ldots n}$ occurring? We need to find out the sum of all the samples with a probability $P = > P_{1\ldots n}$†.

* Bayes' hypothesis of the rectangular distribution of probabilities, although it clearly does not apply to all probabilities, certainly does apply to all probability integral values.

† The reader must assure himself at this point that we are adopting methods of treatment with which he is already familiar. That is to say, we are extending the argument he is accustomed to use with regard to the probability integral for a single value x_1; for if p_1 be extremely small, he argues that it is reasonable to suppose that x_1 was not a random sample from the curve he has used to compute its probability integral. Now p_1 is the chance that a certain variate x_1 will not exceed a value ξ_1, p_2 that a second variate x_2 will not exceed a value ξ_2 and so on. Then, since $x_1, x_2, \ldots x_n$ are supposed to be independent random samples, it follows that $P_{1\ldots n} = p_1 p_2 \ldots p_n$ is the chance that the combined system $x_1, x_2, \ldots x_n$ will not exceed the system $x_1, x_2, \ldots x_n$. We then turn the problem round so to speak and ask what other series of $x_1, x_2, \ldots x_n$ will give a probability exceeding $P_{1\ldots n}$. That is to say, we divide up the possible field of $p_1, p_2, \ldots p_n$'s by a contour surface which separates the sub-field of $p_1, p_2, \ldots p_n$'s which gives less combined probabilities from the sub-field which gives greater probabilities than $P_{1\ldots n}$. If the sub-field of less probabilities be very small, then we judge that it is unreasonable to suppose the observed $x_1, x_2, \ldots x_n$ form a random sample from the system or systems of distribution

To answer this we must first solve the following problem:

We have a system of n rectangular axes in Euclidean space and a point $z_1, z_2, \dots z_n$. This point is constrained to lie in the "n-cuboid" with all its edges equal to unity. A surface is given by the equation

$$\lambda_n = z_1 z_2 \dots z_n \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)\,bis,$$

which may be termed the "n-hyperboloid," we require to find out what volume of the n-cuboid is cut off by the n-hyperboloid.

Let us denote the volume of the n-cuboid *inside* the hyperboloid by V_n. Let us assume that

$$V_n = 1 - \lambda_n \left(1 - \frac{\log_e \lambda_n}{1!} + \frac{(\log_e \lambda_n)^2}{2!} - \frac{(\log_e \lambda_n)^3}{3!} + \dots + (-1)^{n-1} \frac{(\log_e \lambda_n)^{n-1}}{(n-1)!}\right) \quad \dots\dots(2).$$

Now add another variate z_{n+1}, so that

$$z_1 z_2 z_3 \dots z_n z_{n+1} = \lambda_{n+1}.$$

If z_{n+1} be made constant the section of the $(n+1)$-hyperboloid is an n-hyperboloid and the "volume" of this is given by (2) above. We must integrate these strips of "volume" from $z_{n+1} = \lambda_{n+1}$ to 1 and accordingly

$$V_{n+1} = \int_{\lambda_{n+1}}^{1} V_n \, dz_{n+1}$$

$$= \int_{\lambda_{n+1}}^{1} \left[1 - \frac{\lambda_{n+1}}{z_{n+1}} \left(1 - \frac{\log_e \frac{\lambda_{n+1}}{z_{n+1}}}{1!} + \frac{\left(\log_e \frac{\lambda_{n+1}}{z_{n+1}}\right)^2}{2!} - \dots + (-1)^{n-1} \frac{\left(\log_e \frac{\lambda_{n+1}}{z_{n+1}}\right)^{n-1}}{(n-1)!}\right)\right] dz_{n+1}.$$

Suppose

$$y = \log_e \frac{\lambda_{n+1}}{z_{n+1}},$$

then

$$V_{n+1} = \left[z_{n+1}\right]_{\lambda_{n+1}}^{1} + \lambda_{n+1} \int_0^{\log_e \lambda_{n+1}} \left(1 - \frac{y}{1!} + \frac{y^2}{2!} - \dots + \frac{(-1)^{n-1} y^{n-1}}{(n-1)!}\right) dy$$

$$= 1 - \lambda_{n+1} + \lambda_{n+1} \left(\frac{\log_e \lambda_{n+1}}{1!} - \frac{(\log_e \lambda_{n+1})^2}{2!} + \frac{(\log_e \lambda_{n+1})^3}{3!} - \dots + \frac{(-1)^{n-1}}{n!} (\log_e \lambda_{n+1})^n\right)$$

$$= 1 - \lambda_{n+1} \left(1 - \frac{\log_e \lambda_{n+1}}{1!} + \frac{(\log_e \lambda_{n+1})^2}{2!} - \dots + \frac{(-1)^n}{n!} (\log_e \lambda_{n+1})^n\right),$$

upon which we have based the calculation of their probability integrals. We do not assert that our test is the most stringent test, but that it is a very general and often easily applied systematic test of a given hypothesis. It may, indeed, be doubted whether any test is the most stringent throughout the whole area in which it has become customary to apply it. We may say of most tests, as of the present, that they may disprove any hypothesis, but that if they merely render a reasonable probability for the hypothesis, we cannot be certain that a more stringent test may not exist which would render the hypothesis very unlikely, or that a second hypothesis may not have a still greater probability.

or, if (2) holds for V_n, it will hold for V_{n+1}. But for $n=2$, $z_1 z_2 = \lambda_2$, and clearly

$$V_2 = \int_{\lambda_2}^{1} \left(1 - \frac{\lambda_2}{z_1}\right) dz_1 = 1 - \lambda_2 - \lambda_2 \left[\log_e z_1\right]_{\lambda_2}^{1}$$
$$= 1 - \lambda_2 (1 - \log_e \lambda_2).$$

Hence the formula holds for V_2 and will therefore hold for V_3 and so on by induction, and is thus generally true.

We see accordingly that

$$V_n = 1 - \lambda_n \times \text{sum of first } n \text{ terms of the exponential series for } e^{-\log_e \lambda_n}.$$

But the first n terms of the exponential series e^x

$$= e^x (1 - I(n-1, x)),$$

where I denotes the incomplete Γ-function ratio. Thus

$$V_n = 1 - \lambda_n e^{-\log_e \lambda_n} (1 - I(n-1, -\log_e \lambda_n))$$
$$= I(n-1, -\log_e \lambda_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Since λ_n must always be less than unity the incomplete Γ-function ratio is real. If we use the *Tables of the Incomplete* Γ-*Function* *, we must look out $I(n-1, u)$, where

$$u = -\frac{\log_e \lambda_n}{\sqrt{n}} = -\frac{\log_{10} \lambda_n}{\sqrt{n} \log_{10} e} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now let us return to our original problem. We have a sample of n variates $x_1, x_2, \ldots x_n$ taken from a population following a given or supposed law of distribution of which we know the probability integral. The values of this for our n sample variates $x_1, x_2, \ldots x_n$ are respectively $p_1, p_2, \ldots p_n$. These probabilities follow a rectangular distribution. If we take $\lambda_n = p_1 p_2 \ldots p_n$—the probability of the occurrence of the particular independent set of probabilities $p_1, p_2, \ldots p_n$—then the probability P_{λ_n} of a combination occurring with a probability value as great as or greater than λ_n is given by

$$P_{\lambda_n} = I\left(n-1, -\frac{\log_{10} \lambda_n}{\sqrt{n} \log_{10} e}\right),$$

or if Q_{λ_n} be the probability of a lower probability occurring,

$$Q_{\lambda_n} = 1 - P_{\lambda_n} = 1 - I\left(n-1, -\frac{\log_{10} \lambda_n}{\sqrt{n} \log_{10} e}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $I(p, u)$ is the function tabled in the *Tables of the Incomplete* Γ-*Function*.

The effectiveness of this method of approaching the problem of small random samples lies in the facts, (i) that grouping the individuals of small samples to obtain a (P, χ^2) test is a somewhat hazardous proceeding when n is small, (ii) we do not

* H.M. Stationery Office, 1922.

make the somewhat rash assumption that binomials $(p+q)^s$ in which p is not nearly equal to q, and s is small, may be replaced by a normal curve.

Given that the probability integral of the supposed (or real) parent population is known, then we find Q_{λ_n} without any approximations or hypotheses.

If Q_{λ_n} be very small, we have obtained an extremely rare sample, and we have then to settle in our minds whether it is more reasonable to suppose that we have drawn a very rare sample at one trial from the supposed parent population, or that our hypothesis as to the character of the parent population is erroneous, i.e. that the sample $x_1, x_2, \ldots x_n$ was not drawn from the supposed population.

This "settling of what is reasonable" occurs not only with the (P, χ^2) test*, but with the application of most criteria in statistics. It is not peculiar to the present, or, as I propose to call it, the (P, λ_n) test.

A table of P_{λ_n} for $n=2$ to 30 by units and $-\log_{10}\lambda_n$ from 0 upwards by ·125 or ·250 can be formed from the *Incomplete* Γ-*Function Tables*, and when published will expedite the application of the (P, λ_n) test. Such a table is completed, and will shortly be published.

Another very important attribute of this (P, λ_n) test may now be mentioned. Let us suppose we have any number of parent populations each following its own law of distribution, then if x_s be an individual drawn *at random* from the sth population, its probability integral will be also a random sample from a rectangular distribution. Hence the individuals $x_1, x_2, \ldots x_n$ need not be drawn from the same parent population, but may be drawn from any number of populations, one or more from each. Their probability integrals $p_1, p_2, \ldots p_n$ will all be random samples from a rectangular distribution and may be combined to serve as a random sample of n from such a distribution. Our (P, λ_n) test is accordingly a test of randomness, and not a test of whether the series of individuals have been drawn from a single particular type of distribution—z-distribution, σ^2 distribution, normal distribution or what not. If the sample of n gives a highly improbable result, then we doubt its randomness. This want of randomness may arise because the selection of them or a certain number of them has not been random, or because their probability integrals or some of them have been calculated on the basis of a hypothesis as to their parent population or parent populations which is in itself incorrect, e.g. a sample of x's might have their probability integrals calculated from a normal curve, whereas in fact they were really a random sample from a Type I curve. The probability integrals would not then appear as a reasonable random sample from a rectangular distribution. When we conclude that our sample of probability integrals is very improbably random, we must turn to other sources to determine whether it is owing to the

* There exists a relation between the (P, χ^2) and (P, λ_n) tests for which, I think, some explanation must be forthcoming. Namely, P_{χ^2} can be found from the $I(p, u)$ Tables by taking

$$1 - P_{\chi^2} = I\left(\tfrac{1}{2}(n'-3), \frac{\frac{1}{2}\chi^2}{\sqrt{\frac{1}{2}(n'-1)}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Hence if we take out the P_{χ^2} corresponding to $n'=2n+1$ and $\chi^2 = -2\log_{10}\lambda_n/\log_{10}e$, then P_{χ^2} will equal Q_{λ_n}; and thus within their range the (P, χ^2) Tables may be used to find Q_{λ_n}.

selection being biased, or to an erroneous hypothesis as to the parental distributions on the basis of which we have computed our probability integrals*.

As a result of the above we see that this new (P, λ_n) method enables us to form *combined* tests, some of the x's may be means, others standard deviations, others correlation coefficients, etc. etc. All we need do, if these quantities are uncorrelated, is to calculate their probability integrals from the appropriate distributions using when necessary the most probable values, as determined from the samples, of the constants required in the parental distributions. Illustrations of such combined tests are given below.

If we suppose the probability integrals, p_s $(s = 1, 2, \ldots n)$, to be uniformly distributed, for example, by dividing the range 0 to 1 into n equal sections, and placing a value of p_s in the centre of each of them, so that p_s takes the value $(2s-1)/2n$, then the corresponding χ^2 would be zero, and $P_{\chi^2} = 1$. There is something difficult about this. The result agrees with the mathematical expectation, "one ball in each compartment," but the basis of the *theory* is really too narrow†. The probability of the result must depend upon *where* the p's fall in each compartment. Let us consider what happens in the corresponding case of P_{λ_n}.

We have

$$e^{-\sqrt{n}u} = \lambda_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(2n)^n} = \frac{(2n)!}{n!\, 2^{2n} n^n}$$

* This dilemma as to randomness occurs as far as I can see in most published tests, e.g. we assume as hypothesis that the two samples were drawn from the same normal population, and find it very improbable, this may really be due to bias in the taking of one or other sample and not to the absence of normality. Only further knowledge or investigation can lead to a discrimination between the two possibilities. For example, we have the means and standard deviations of two samples, and we wish to ascertain whether their parent populations are differentiated. We adopt the z-test. In doing so we make two hypotheses, (i) that of normal distributions, (ii) that both samples are truly random. Hence we have at least three possibilities. (*a*) that one or both parent populations have not normal distributions, (*b*) that one or both samples are not random and (*c*) that the two samples do not come from the *same* normal population. It seems to me that many users of the test, when they get an improbable result assume straight off that (*c*) must be the origin of it, and do not question the possibility of (*a*) or (*b*) being the source of the observed improbability. The P_{λ_n} test is no worse (or no better) than the other tests in this respect, i.e. that we have to consider whether the observed want of randomness in the distribution of the probability integrals is due to the hypothesis as to the nature of the distribution curve or to want of the other randomness in the process of sampling. It is unfortunate that the same word "randomness" has to be used in two places in the same investigation. The want of "randomness" in the distribution of probability integrals may or may not be evidence of want of randomness in the sampling.

† Another point is common to other tests as well as our present test, although at times the point is overlooked, namely a result may—to speak paradoxically—be so highly probable as to be wholly improbable. My memory carries me back many years to a memoir I read in manuscript. It concerned the distribution of a character in over a thousand offspring of certain sets of parents; the character being treated as involving five or six Mendelian factors. The classified offspring distributed themselves absolutely accurately according "to the mathematical expectation"; but the odds of course against such an occurrence were immense, and this was pointed out to the writer. The memoir afterwards appeared in print, the "improbable high probability" having disappeared. This development not only convinced me of the elasticity of Mendelian categories, but led me to realise how great is the risk that a biologist, ignorant of mathematical statistics, may heedlessly run in his enthusiasm for a hypothesis. His results may be "too good to be true."

or $$u = \frac{\sqrt{n}}{\log_{10} e}\left(\frac{1}{n}\log_{10}(n!) + \log_{10} n + 2\log_{10} 2 - \frac{1}{n}\log_{10}((2n)!)\right) \quad \ldots\ldots\ldots(7).$$

Applying Sterling's Theorem to the value of $e^{-\sqrt{n}u}$, we find for the asymptotic value of u,

$$u_{n\to\infty} = \sqrt{n} - \frac{\log 2}{2\sqrt{n} \times \cdot 434{,}2945} \quad \ldots\ldots\ldots(8).$$

Accordingly we have the following results, the first two being obtained by linear interpolation from the *Tables of the Incomplete* Γ-*Function* and liable to an error of a unit in the last decimal of P_{λ_n}.

n	u	$I(n-1, u)$	P_{λ_n}
5	2·08479	$I(4, 2{\cdot}08479)$	·5018
10	3·05400	$I(9, 3{\cdot}05400)$	·5008
100	9·96561	$I(99, 9{\cdot}96561)$	·5004
500	22·34523*	$I(499, 22{\cdot}34523)$	·5002
1000	31·62763†	$I(999, 31{\cdot}62763)$	·5001
∞	∞	$I(\infty, \infty)$	·5000

Thus we see that P_{λ_n} rapidly approaches the value ·5; in other words, in an absolutely uniform distribution of the probability integrals p_s there would be as many distributions above as below this value. This is what we might *a priori* expect, but it is suggestive in endeavouring to interpret the relation of P_{λ_n} to P_{χ^2}. We will return to this point shortly.

In order to determine the values of the incomplete Γ-function for 99, 499 and 999, recourse was had to the E method of p. xvi of the *Instructions as to the use of the Tables*‡. This method is peculiarly advantageous in the present case both practically and theoretically.

If $p = n-1$, and $v = \sqrt{n}\,u = \sqrt{p+1}\,u = p + d'$,

$$\int_0^v \frac{x^p e^{-x} dx}{\Gamma(p+1)} = 0{\cdot}5 + y_2(d'-d) + \frac{9y_1 - 185y_2 + 180y_3 - 4y_4}{540}(d'^3 - d^3)$$
$$- \frac{6y_1 + 10y_2 - 15y_3 - y_4}{360}(d'^4 - d^4) + \frac{3y_1 + 25y_2 - 30y_3 + 2y_4}{900}(d'^5 - d^5) \quad \ldots(9),$$

where $$y_1 = \frac{(p-2)^p e^{-(p-2)}}{\Gamma(p+1)}, \quad y_2 = \frac{p^p e^{-p}}{\Gamma(p+1)},$$
$$y_3 = \frac{(p+1)^p e^{-(p+1)}}{\Gamma(p+1)}, \quad y_4 = \frac{(p+3)^p e^{-(p+3)}}{\Gamma(p+1)},$$

and $$d = \cdot 6666{,}6667 + \frac{\cdot 0197{,}5309}{p+1} + \frac{\cdot 0072{,}1144}{(p+1)^2} + \frac{\cdot 0003{,}8554}{(p+1)^3} \quad \ldots\ldots(10)$$

gives the distance from mode to median.

* Differs by five units in the last figure from the asymptotic value.

† Asymptotic value.

‡ *Tables of the Incomplete* Γ-*Function.*

Working to four figures only the first two or three terms in (9) and the first two in (10) suffice when p is of the order 100 and more, and the values of the y's are easily determined. Now $v = p + d'$ and the asymptotic value of

$$v = \sqrt{n}\,u = \sqrt{n}\left(\sqrt{n} - \frac{\log 2}{2\sqrt{n} \times \cdot 434{,}2945}\right)$$

$$= n - 1 + \left(1 - \frac{\log 2}{\cdot 868{,}5890}\right) = p + \cdot 653{,}4264.$$

Thus $d' - d =$ a very small quantity, i.e. approaches ·01324, and since y_2 is of the order ·01 to ·02 even the second term of (9) only influences the fourth decimal place, and we see how the approximation to ·5000 arises.

Now while the value of P_{χ^2} remains the same wherever we place the p_s in the n divisions, the value of P_{λ_n} will vary according to their position in those divisions, and it seems very proper that it should. We will consider three cases of the p's for $n = 5$. Instead of taking the values $p_1 = \cdot 1$, $p_2 = \cdot 3$, $p_3 = \cdot 5$, $p_4 = \cdot 7$ and $p_5 = \cdot 9$ we will:

(A) Give the p's larger values close up to the boundaries of the divisions, namely: $p_1 = \cdot 19$, $p_2 = \cdot 39$, $p_3 = \cdot 59$, $p_4 = \cdot 79$ and $p_5 = \cdot 99$.

(B) Give the p's smaller values close up to the opposite boundaries of the five divisions, namely: $p_1 = \cdot 01$, $p_2 = \cdot 21$, $p_3 = \cdot 41$, $p_4 = \cdot 61$ and $p_5 = \cdot 81$.

(C) Take values drawing up the p's closely to the central value ·50, namely: $p_1 = \cdot 19$, $p_2 = \cdot 39$, $p_3 = \cdot 50$, $p_4 = \cdot 61$, $p_5 = \cdot 81$.

We have $\sqrt{n} \log_{10} e = \cdot 971{,}1120$, and accordingly

(A) $\log \lambda_n = -1 \cdot 466{,}0675$, $\quad u = 1 \cdot 5096$, $\quad Q_{\lambda_n} = \cdot 7486$,

(B) $\log \lambda_n = -3 \cdot 371{,}1820$, $\quad u = 3 \cdot 47147$, $\quad Q_{\lambda_n} = \cdot 1142$,

(C) $\log \lambda_n = -1 \cdot 737{,}3970$, $\quad u = 1 \cdot 78908$, $\quad Q_{\lambda_n} = \cdot 6287$.

Now we notice that while $P_{\chi^2} = 1$ for all these cases*, Q_{λ_n}—while in none of the three cases giving an improbable result—is far from giving identical measures for the three distributions of the p's. When we move the central points towards the greater values of p_s, i.e. in (A), we find that 75 °/₀ of cases have a less degree of probability; when we move them towards the lesser degrees of probability, i.e. in (B), we find that only about 11 °/₀ have a lesser degree of probability, and finally when we endeavour to concentrate towards the centre of the entire range, i.e. in (C), we find the intermediate value, namely about 63 °/₀ of cases have a lesser probability.

This illustration will I think suffice to indicate some advantages of the P_{λ_n} test over the P_{χ^2} test.

Illustration 1. *Use of the Probability Integral from the Table of the Normal Curve.*

The mean length of life of 15 samples of five electric lamps is provided by E. S. Pearson.

* This is quite apart from the questionable replacing of a binomial by a normal curve in such a case as $(\frac{1}{2} + \frac{1}{2})^5$.

TABLE I.

Length of Life of Lamps in Hours.*

Sample No.	Mean	Standard Deviation	Sample No.	Mean	Standard Deviation
1	1295	440	9	1715	385
2	2005	435	10	1650	460
3	2445	580	11	1935	560
4	1900	345	12	1760	280
5	2570	290	13	2175	465
6	1980	510	14	1570	505
7	1990	445	15	1670	380
8	1990	315	—	—	—

Is it reasonable to suppose that these 15 five lamp means have been drawn from the same normal population?

In order to answer this question we must determine what are the most probable values to assign to the mean M and the standard deviation Σ of this supposed common parent population. Let the samples be v in number, and of different sizes n_t, the mean and standard deviation of the tth sample being m_t and s_t. Then the most probable values of M and Σ^2 will be obtained by making the following expression a maximum:

$$E = \prod_{t=1}^{v}(n_t)\,\frac{1}{(\sqrt{2\pi})^v\,\Sigma^v}\,e^{-S_{t=1}^{v}\frac{1}{2}\frac{n_t(m_t-M)^2}{\Sigma^2}} \times \frac{1}{2^{N-2v}}\prod_{t=1}^{t=v}\frac{1}{\Gamma(\frac{1}{2}(n_t-2))}$$

$$\times \frac{1}{\Sigma/\sqrt{2n_t}}\left(\frac{s_t}{\Sigma/\sqrt{2n_t}}\right)^{n_t-2} e^{-\frac{1}{2}\left(\frac{s_t}{\Sigma_t/\sqrt{2n_t}}\right)^2} \quad \ldots\ldots(11).$$

Taking the logarithmic differentials with regard to M and Σ we find

$$S_{t=1}^{t=v}\, n_t(m_t - M)/\Sigma^2 = 0,$$

or
$$M = S_{t=1}^{t=v}\frac{(n_t m_t)}{N}, \text{ where } N = S(n_t) \quad \ldots\ldots(12),$$

thus M is the weighted mean of the sample means. Further, after differentiating and collecting terms we find

$$\Sigma^2 = \frac{S n_t(s_t^2 + (m_t - M)^2)}{N} \quad \ldots\ldots(13).$$

Clearly (12) and (13) simply amount to saying that the best value to give M and Σ is that obtained by pooling all the individual values in the samples and finding the mean and standard deviation of the combination. We may note here that if our

* *Tables of the Incomplete* B-*Function*, Introduction, p. lii.

hypothesis were that our samples were drawn from independent parent populations, but with the same variability, we should have to replace the M in (13) by M_t and differentiate with respect to every M_t. This gives us $M_t = m_t$, and

$$\Sigma^2 = \frac{Sn_t(s_t^2)}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

i.e. the weighted square of the variances. The hypothesis that M_t varies from sample to sample may indicate, however, a secular change in M_t and so involve correlation between successive samples, which has not been allowed for in (11) where the samples are considered independent.

We proceed to find M and Σ for the data of the electric lamps. The needful calculations are provided in Table I. If we have determined M and Σ, then the m's are distributed normally with a mean M and standard deviation $\Sigma/\sqrt{n}$, so that we have only to look up $(M - m_t)/(\Sigma/\sqrt{n})$ in the table of the normal probability integral to obtain the series of probability integrals, p_t, in column (h) of our table. Great care must be taken to regard the sign of $(M - m_t)/(\Sigma/\sqrt{n})$ as the areas must all be measured from *one* end of the normal curve.

TABLE II.

(*a*) Sample No.	(*b*) Mean	(*c*) $m_t - M$	(*d*) $(m_t - M)^2$	(*e*) s_t	(*f*) s_t^2	(*g*) $(c)/(\Sigma/\sqrt{n})$	(*h*) p_t	(*i*) $\log_{10} p_t$
1	1295	−615	378225	440	193600	−2·555	·00052	$\bar{4}$·716,0033
2	2005	+ 95	9025	435	189225	+ ·395	·65536	$\bar{1}$·816,4799
3	2445	+535	286225	580	336400	+2·222	·98686	$\bar{1}$·994,2555
4	1900	− 10	100	345	119025	− ·042	·48352	$\bar{1}$·684,4144
5	2570	+660	435600	290	84100	+2·742	·96946	$\bar{1}$·986,5299
6	1980	+ 70	4900	510	260100	+ ·291	·61447	$\bar{1}$·788,5007
7	1990	+ 80	6400	445	198025	+ ·332	·63005	$\bar{1}$·799,3750
8	1990	+ 80	6400	315	99225	+ ·332	·63005	$\bar{1}$·799,3750
9	1715	−195	38025	385	148225	− ·810	·20897	$\bar{1}$·320,0839
10	1650	−260	67600	460	211600	−1·080	·14007	$\bar{1}$·146,3451
11	1935	+ 25	625	560	313600	+ ·104	·54141	$\bar{1}$·733,5263
12	1760	−150	22500	280	78400	− ·623	·26664	$\bar{1}$·425,9253
13	2175	+265	70225	465	216225	+1·101	·86455	$\bar{1}$·936,7901
14	1570	−340	115600	505	255025	−1·412	·12631	$\bar{1}$·101,4377
15	1670	−240	57600	380	144400	− ·997	·15963	$\bar{1}$·203,1145
$\frac{1}{15}$ sum = M = 1910		$\frac{1}{15}$ sum = 99937		$\frac{1}{15}$ sum = 189,812		—	log λ_n = − 8·547,8434	

$\Sigma^2 = 99{,}937 + 189{,}812 = 289{,}749$, $\sqrt{n}\log_{10} e = 1{\cdot}682{,}0153$, $\Sigma = 538{\cdot}28$, and $\Sigma/\sqrt{n} = 240{\cdot}7261$.

Thus $u = 5{\cdot}081{,}906$, and we require $I(14,\ 5{\cdot}081{,}906)$; we find this from the *Tables of the Incomplete* Γ-*Function* to be ·88209. Hence $Q_n = {\cdot}118$: or, the number of more improbable sets of 15 samples is 11·8 %.

We cannot on the result of this test assert that the lamps' lives were certainly not samples from the same population.

Illustration 2. *Use of the Incomplete* B-*Function Table to determine the Probability Integrals.* We may use the data of Illustration 1 to exemplify this method. "Student" deduced* that if M was the mean of the parent population and m_t the mean, s_t the standard deviation of a sample of size n_t, then the distribution of $z_t = (m_t - M)/s_t$ is given by the curve

$$y = y_0 \frac{1}{(1+z_t^2)^{\frac{1}{2}n_t}} \quad \text{.....(15).}$$

We require the probability integral of this curve, and various tables have been computed for it, either for z_t or some modified form of z_t. I personally have found nothing so comprehensive and convenient as the *Incomplete* B-*Function Table.*

The probability integral is

$$p(n, z) = \int_{-\infty}^{z} \frac{dz}{(1+z^2)^{\frac{1}{2}n}} \Big/ \int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)^{\frac{1}{2}n}}$$

$$= \cdot 5 + \int_0^z \frac{dz}{(1+z^2)^{\frac{1}{2}n}} \Big/ \left(2 \int_0^{\infty} \frac{dz}{(1+z^2)^{\frac{1}{2}n}} \right).$$

Put $$z^2 = \frac{x}{1-x} \quad \text{or} \quad x = \frac{z^2}{1+z^2},$$

then $$p(n, z) = \cdot 5 \left\{ 1 + \int_0^z x^{-\frac{1}{2}} (1-x)^{\frac{1}{2}(n-3)}\, dx \Big/ \int_0^1 x^{-\frac{1}{2}} (1-x)^{\frac{1}{2}(n-3)}\, dx \right\}.$$

But the ratio of the two integrals $= I_x(\frac{1}{2}, \frac{1}{2}(n-1))$ or the incomplete B-function ratio, the quantity $I_x(p, q)$ tabled in the corresponding book of tables. Thus we may write

$$\begin{aligned} p(n, z) &= \cdot 5 \{1 + I_x(\tfrac{1}{2}, \tfrac{1}{2}(n-1))\} \\ &= \tfrac{1}{2} \{2 - I_{1-x}(\tfrac{1}{2}(n-1), \tfrac{1}{2})\} \\ &= \tfrac{1}{2} \{2 - I_{\frac{1}{1+z^2}}(\tfrac{1}{2}(n-1), \tfrac{1}{2})\} \end{aligned} \quad \text{.....(16).}$$

The reason for this change of expression is that the *Tables of the Incomplete* Γ-*Function* are provided for $p \geqslant q$. We must remember, however, as especially important that we have got to allow for z being negative. Accordingly we take

$$\left.\begin{aligned} p(n, +z) &= 1 - \tfrac{1}{2} I_{\frac{1}{1+z}}(\tfrac{1}{2}(n-1), \tfrac{1}{2}) \\ p(n, -z) &= \tfrac{1}{2} I_{\frac{1}{1+z^2}}(\tfrac{1}{2}(n-1), \tfrac{1}{2}) \end{aligned}\right\} \quad \text{.....(17).}$$

For present purposes linear interpolation into the B-function table will suffice. The following table indicates the needful work. M remains = 1910.

* *Biometrika*, Vol. VI. p. 8.

TABLE III.

Sample No.	$m_t - M$	s_t	z_t	$1+z_t^2$	$1-x_t = 1/(1+z_t^2)$	p_{z_t}	$\log p_{z_t}$
1	−615	440	−1·39772	2·95362	·33856	·0021	$\bar{3}$·322,2193
2	+ 95	435	+0·21839	1·04769	·95448	·7089	$\bar{1}$·850,5850
3	+535	580	+0·92241	1·85084	·54030	·9776	$\bar{1}$·990,1612
4	− 10	345	−0·02899	1·00084	·99916	·4915	$\bar{1}$·691,5235
5	+660	290	+2·27586	6·17954	·16182	·9997	$\bar{1}$·999,8697
6	+ 70	510	+0·13725	1·01884	·98151	·6355	$\bar{1}$·803,1156
7	+ 80	445	+0·17978	1·03232	·96869	·6753	$\bar{1}$·829,4967
8	+ 80	315	+0·25397	1·06456	·93940	·7384	$\bar{1}$·868,2917
9	−195	385	−0·50649	1·25653	·79584	·1111	$\bar{1}$·045,7141
10	−260	460	−0·56522	1·31947	·75788	·0893	$\bar{2}$·950,8515
11	+ 25	560	+0·04464	1·00199	·99801	·5201	$\bar{1}$·716,0869
12	−150	280	−0·53571	1·28698	·77701	·0997	$\bar{2}$·998,6925
13	+265	465	+0·56989	1·32477	·75485	·9123	$\bar{1}$·960,1377
14	−340	505	−0·67327	1·45329	·68809	·0590	$\bar{2}$·770,8520
15	−240	380	−0·63158	1·39889	·71485	·0693	$\bar{2}$·840,7332

As before $\sqrt{n}\log_{10}e = 1{\cdot}682{,}0153$, $S\log p_{z_t} = -9{\cdot}361{,}6694 = \log\lambda_n$.

Thus $u = 5{\cdot}565746$, and $Q_{\lambda_n} = 1 - I\,(14,\ 5{\cdot}56575)$,

where the latter function is the Γ-function ratio. Interpolating linearly from the *Incomplete* Γ-*function Table*, we have

$$Q_n = 1 - {\cdot}93041 = {\cdot}0696.$$

Thus some 7 °/₀ of series of sample would provide a greater degree of improbability.

Dr Egon S. Pearson has applied to the same data in the Introduction to the *Tables of the Incomplete* B-*Function* a third test, namely a modification of Fisher's test for determining whether the regression of one character on a second is a horizontal straight line. This involves the computing of $\eta^2 = \sum_{t=1}^{v} n_t \frac{(m_t - M)^2}{\Sigma^2}$, and the determination of $I_{1-\eta^2}(\frac{1}{2}(N-v), \frac{1}{2}(v-1))$ from the *Tables of the Incomplete* B-*Function*. Thus as in our *Illustration* 1 we have to determine the most probable values of M and Σ, but there is only one value to be found from the table, not 15 from that table, and one from the *Incomplete* Γ-*function Table*. In this case $\eta^2 = {\cdot}3449$ and we have $Q_{\lambda n} = I_{\cdot 6551}(30,\ 7) = {\cdot}0152$, or there are 1·52 °/₀ of cases only more improbable. It will be seen that this result is more stringent than either of our previous tests. Can we find a reason for this? Fisher's test is based on a triple hypothesis: (i) linearity of regression, (ii) homoscedasticity of arrays and (iii) their normality. Now the approach to a normal distribution of means is fairly rapid even for distributions not absolutely normal. Further the formulae (12) and (13) for M and Σ^2, while we have deduced them from a normal distribution, are most reasonable formulae to take in non-normal distributions, and lastly no assumption is made as to homoscedasticity of arrays. Accordingly we might

anticipate that our test in *Illustration* 1 would cover a wider range than Fisher's triple hypothesis does, and so it appears from the result.

Again in the z-test of the present *Illustration* we do suppose that each sample is taken from a normal distribution, and that all these normal parent populations have the same mean, but we do not insist on the standard deviations of all these parent populations being the same. The probability integrals of the several z_t can be combined, as they are random selections of p_t between 0 and 1. Thus again the hypotheses involved do not seem so stringent as in Fisher's case.

With regard to the comparison of the tests in *Illustrations* 1 and 2, it is not at once obvious why the selection from v normal populations having the same means but not *necessarily* the same standard deviations should be more stringent than the selection from v populations having the same means and standard deviations but not *necessarily* strictly normal.

Illustration 3. In an experiment* in which a certain number of children were given raw milk for four months and the same number of children of closely the same age, stature and weight were given pasteurised milk for the same time, the following system of mean growth differences, standard deviations of those differences and the ratios (z's) of those differences to their standard deviations were obtained. The numbers are sufficiently large to admit of our computing the probability integral of z from the normal curve table:

TABLE IV.

Boys. Central Age in years	No. of Pairs	Raw Mean—Pasteurised	Standard Deviation of Difference	z_t	Probability Integral of $z_t = p_t$	$\log_{10} p_t$
$6\frac{1}{2}$	73	−·066	·054	−1·22	·888,7676	$\bar{1}$·948,7882
$7\frac{1}{2}$	76	+·022	·053	+0·41	·340,9030	$\bar{1}$·532,6308
$8\frac{1}{2}$	71	−·003	·052	−0·06	·523,9222	$\bar{1}$·719,2669
$9\frac{1}{2}$	77	+·011	·055	+0·20	·420,7403	$\bar{1}$·624,0141
$10\frac{1}{2}$	60	+·002	·057	+0·04	·484,0466	$\bar{1}$·684,8872
		(*The units are inches.*)			Total	$\bar{2}$·509,5872

Thus: $\log_{10} \lambda_5 = 1{\cdot}490{,}4128$, $\sqrt{n} \log_{10} e = \sqrt{5} \times 434{,}2945 = {\cdot}971{,}1120$ and accordingly $u = 1{\cdot}53475$.

The volume within the 5-hyperboloid $= I(4, 1{\cdot}53475) = {\cdot}262$, or the probability of a set of values of z with a less probability than those observed is ·738. That is to say that if the system of z's were really drawn from normal populations about 74 % of the cases would be less probable. We cannot accordingly assert that there is any difference in growth in these boys according as to whether they took raw or pasteurised milk.

* E. M. Elderton, "The Lanarkshire Milk Experiment," *Annals of Eugenics*, Vol. v. pp. 326—336.

Illustration 4. In the experiment referred to in the previous *Illustration* the following results were obtained in the same manner for the Weight of Girls when Raw Milk was administered and when no milk was given, the pairs being taken of closely the same Age, Stature and Weight.

TABLE V.

Girls. Central Age in years	No. of Pairs	Raw Milk—Control	Standard Deviation of Difference	z_t	Probability Integral of $z_t = p_t$	$\log_{10} p_t$
$6\frac{3}{4}$	144	+ 0·13	2·02	·06	·476,0778	$\bar{1}$·677,6799
$7\frac{3}{4}$	128	+ 1·12	2·41	·46	·322,7581	$\bar{1}$·508,8771
$8\frac{3}{4}$	133	+ 7·98	2·66	3·00	·001,3499	3·130,3016
$9\frac{3}{4}$	133	+ 5·62	2·77	2·03	·021,1783	$\bar{2}$·325,8911
$10\frac{3}{4}$	115	+11·66	3·27	3·57	·000,1785	4·251,6382
		(*The units are ounces.*)			Total	$\bar{10}$·894,3879

Thus: $\log_{10} \lambda_5 = +9{\cdot}105{,}6121$, $\sqrt{n} \log_{10} e = {\cdot}971{,}1120$ and accordingly $u = 9{\cdot}37648$, and the volume inside the hyperboloid $= I(4, 9{\cdot}37648) = {\cdot}999{,}9922$, or the probability of a set of z's occurring with as great as, or a greater improbability than this set is only ·000,0078. We should accordingly argue that the raw milk feeders and the controls can only be, with the highest degree of improbability, random samples of the same population, i.e. the raw milk accelerated the growth of the girls (especially the elder girls) in weight.

Illustration 5. *Use of Probability Integral of Correlation Coefficient.* A number of coefficients of correlation, $r_1, r_2, \ldots, r_t, \ldots r_u$, are found from samples of sizes $n_1, n_2, \ldots n_t, \ldots n_u$. The corresponding means and standard deviations for the samples, the variates being x and y, are $\bar{x}$, $\bar{y}$, σ_{x_1}, σ_{y_1}; $\bar{x}_z$, $\bar{y}_z$, σ_{x_2}, σ_{y_2}; $\ldots$; $\bar{x}_t$, $\bar{y}_t$, σ_{x_t}, σ_{y_t}; $\ldots$; $\bar{x}_u$, $\bar{y}_u$, $\sigma_{\bar{x}_u}$, $\sigma_{\bar{y}_u}$. Each correlation may come from a parent population means m_{x_t}, m_{y_t}, standard deviations Σ_{x_t}, Σ_{y_t}, but these parent populations are supposed to have the same value of the correlation coefficient ρ. What is the most probable value to give to ρ, and what is the chance that the populations from which the u samples are drawn really have the same correlation?

We suppose the distributions in the case of each of the u parent populations to be normal, and we will take $Sn_t = N$. The distribution of the tth population, if we suppose x_{t_a}, y_{t_a} to be any member of it, will be

$$z = \frac{M}{2\pi \Sigma_{x_t} \Sigma_{y_t} (1-\rho^2)^{\frac{1}{2}}} e^{-\frac{1}{2(1-\rho^2)} \left\{ \frac{(x_{t_a} - \bar{x}_t)^2}{\Sigma^2_{x_t}} + \frac{(y_{t_a} - \bar{y}_t)^2}{\Sigma^2_{y_t}} - \frac{2\rho (x_{t_a} - \bar{x}_t)(y_{t_a} - \bar{y}_t)}{\Sigma_{x_t} \Sigma_{y_t}} \right\}} \quad \ldots(18).$$

If we take the product of such expressions for all values of a for the tth sample, we find that the chance of such a sample arising from values of x and y lying

between x_{t_a} and $x_{t_a}+\delta x_{t_a}$, y_{t_a} and $y_{t_a}+\delta y_{t_a}$ and similar values for the n_t other pairs of coordinates may be thrown into the familiar form

$$\left(\frac{1}{2\pi\Sigma_{x_t}\Sigma_{y_t}}\right)^{n_t}\frac{1}{(1-\rho^2)^{\frac{1}{2}n_t}} \times e^{-\frac{n_t}{2(1-\rho^2)}\left[\frac{(m_{x_t}-\bar{x}_t)^2+\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{(m_{y_t}-\bar{y}_t)^2+\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho\{(m_{x_t}-\bar{x}_t)(m_{y_t}-\bar{y}_t)+r_t\sigma_{x_t}\sigma_{y_t}\}}{\Sigma_{x_t}\Sigma_{y_t}}\right]}\prod_{a=1}^{a=n_t}(\delta x_{t_a}\delta y_{t_a}) \quad \ldots\ldots(19).$$

We have a similar value for each of the u sets of samples. Now we suppose the u sets of samples to be independent and further nothing known about the constants of the u-parent populations except that they have the same ρ by hypothesis. Accordingly we have to make the product of u expressions like the above, a maximum by choice of the $4u+1$ variates $\bar{x}_t$, $\bar{y}_t$, Σ_{x_t}, Σ_{y_t} $(t=1, 2, \ldots u)$ and ρ. As they are independent, we can differentiate the single values like (18) to determine the values for the first four types of variates, but we must differentiate the combined product to obtain the value of ρ. We have at once $m_{x_t}=\bar{x}_t$, $m_{y_t}=\bar{y}_t$. Inserting these we have to maximise the expression

$$\frac{1}{(1-\rho^2)^{\frac{1}{2}n_t}}\left(\frac{1}{\Sigma_{x_t}\Sigma_{y_t}}\right)^{n_t}e^{-\frac{n_t}{2(1-\rho^2)}\left[\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right]}\prod_{a=1}^{a=n_t}(\delta x_{t_a}\delta y_{t_a}) \quad \ldots(20)$$

to find the proper values of Σ_{x_t} and Σ_{y_t}. If we take the logarithm of this and differentiate with regard to Σ_{x_t} and Σ_{y_t} we find, after dividing out a factor,

$$0=-1+\frac{1}{1-\rho^2}\left(\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}-\frac{\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right),$$

$$0=-1+\frac{1}{1-\rho^2}\left(\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right),$$

and accordingly by subtraction $\dfrac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}=\dfrac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}$; or, since the standard deviations must be positive,

$$\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}=\frac{1-\rho^2}{1-\rho r_t}=\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}} \quad \ldots\ldots(21).$$

We have now to differentiate the product of u expressions like (20) with regard to ρ, where after differentiation we can make use of (21). The required expression as far as ρ is concerned is

$$\frac{1}{(1-\rho^2)^{\frac{1}{2}N}}e^{-\frac{1}{2(1-\rho^2)}\sum_{t=1}^{t=u}n_t\left[\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right]}\prod_{t=1}^{t=u}\prod_{a=1}^{a=n_t}(\delta x_{t_a}\delta y_{t_a}).$$

Assuming that it is allowable to consider the double product independent of Σ_{x_t}, Σ_{y_t} and ρ we have, by taking a logarithmic differential with regard to ρ,

$$0=\frac{N\rho}{(1-\rho^2)^2}-\frac{\rho}{(1-\rho^2)^2}\sum_{t=1}^{t=u}n_t\left[\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right]+\frac{1}{1-\rho^2}\sum_{t=1}^{t=u}\frac{n_t r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}.$$

Now making use of (21) we find, since $\overset{t=u}{\underset{t=1}{S}} n_t = N$,

$$\frac{N\rho}{1-\rho^2} = \overset{t=u}{\underset{t=1}{S}} \left(\frac{n_t r_t}{1-\rho r_t} \right) \quad \text{.....(22).}$$

Our (21) agrees with the (B) equation of Mr Brandner's (11)* and our (22) agrees with his (11) (C), if we take the simple case of $u=2$, when $N = n_1 + n_2$.

Returning to (22), let us write

$$\mu_v = \overset{t=u}{\underset{t=1}{S}} \frac{n_t r_t^v}{N} \quad \text{.....(23),}$$

or, μ_v is the weight mean of the vth powers of the r_t's. Accordingly we may write our equation for the most probable value of ρ in the form

$$\rho = \mu_1 + \rho(\mu_2 - \rho^2) + \rho^2(\mu_3 - \rho^3) + \rho^3(\mu_4 - \rho^4) + \rho^4(\mu_5 - \rho^5) + \dots \quad \text{...(24).}$$

This equation allows us to make rapid approximations to the value of ρ, according as to what power of the correlation coefficients may for a practical purpose be considered negligible. Thus

$$\rho_1 = \mu_1 = \text{mean of the } r_t\text{'s},$$

$$\rho_2 = \mu_1 + \rho_1(\mu_2 - \rho_1^2),$$

$$\rho_3 = \mu_1 + \rho_2(\mu_2 - \rho_2^2) + \rho_1^2(\mu_3 - \rho_1^3),$$

$$\rho_4 = \mu_1 + \rho_3(\mu_2 - \rho_3^2) + \rho_2^2(\mu_3 - \rho_2^3) + \rho_1^3(\mu_4 - \rho_1^4),$$

and so on .....(25).

The following example is taken from the paper by E. S. Pearson and S. S. Wilks in the current issue†:

It is assumed that the frequency of Head Length and Head Breadth in skulls follows a normal distribution. Samples of 20 skulls were taken from 30 different races and their correlations calculated. The dimensions of the skulls and their

TABLE VI.

Racial Correlation Coefficients for equal small Samples of Thirty Races.

Race	r_t	Race	r_t	Race	r_t	Race	r_t	Race	r_t
1	+·097	7	−·037	13	+·319	19	+·018	25	+·245
2	+·198	8	+·667	14	+·310	20	+·160	26	+·360
3	+·576	9	+·014	15	+·019	21	+·178	27	+·592
4	−·015	10	−·112	16	+·445	22	+·763	28	−·515
5	+·173	11	+·219	17	+·410	23	+·101	29	+·023
6	+·764	12	−·152	18	+·946	24	+·449	30	+·254

* *Biometrika*, Vol. xxv. p. 104.

† *Biometrika*, Vol. xxv. pp. 372—374.

standard deviations for the 30 parent populations are supposed unknown, but undoubtedly differ significantly. The problem is whether the correlations based on these small samples can be considered random samples from parent populations having a common correlation coefficient*. It is unnecessary to repeat the names of the races here, as if the problem were to be considered in earnest, we should not take all the n_t's equal to the small number 20, but use all the skulls available and weight the r_t's, using formulae (22) and (23).

The first four moments obtained by adding the powers given in *Barlow's Tables* (Edition, Comrie) are as follows, due attention being paid to the sign of r_t:

$$\mu_1 = \cdot 2489{,}6667, \quad \mu_2 = \cdot 1568{,}5010, \quad \mu_3 = \cdot 0904{,}4030, \quad \mu_4 = \cdot 0677{,}0455.$$

Hence $$\rho_1 = \text{mean}, \quad r_t = \cdot 2489{,}6667.$$

Thus we have

$$\rho_1^2 = \cdot 0619{,}8440, \quad \rho_1^3 = \cdot 0154{,}3205, \quad \rho_1^4 = \cdot 0038{,}4207.$$

$$\begin{aligned}\rho_2 &= \cdot 2489{,}6667 + \cdot 2489{,}6667\,(\cdot 1568{,}5010 - \cdot 0619{,}8440)\\ &= \cdot 2725{,}8507,\end{aligned}$$

whence $$\rho_2^2 = \cdot 0743{,}0262, \quad \rho_2^3 = \cdot 0202{,}5378,$$

$$\begin{aligned}\rho_3 &= \cdot 2489{,}6667 + \cdot 2725{,}8507\,(\cdot 1568{,}5010 - \cdot 0743{,}0262)\\ &\quad + \cdot 0619{,}8440\,(\cdot 0904{,}4030 - \cdot 0154{,}3205)\\ &= \cdot 2761{,}1503,\end{aligned}$$

whence $$\rho_3^2 = \cdot 0762{,}3951,$$

$$\begin{aligned}\rho_4 &= \cdot 2489{,}6667 + \cdot 2761{,}1503\,(\cdot 1568{,}5010 - \cdot 0762{,}3951)\\ &\quad + \cdot 0743{,}0262\,(\cdot 0904{,}4030 - \cdot 0202{,}5378)\\ &\quad + \cdot 0154{,}3205\,(\cdot 0677{,}0455 - \cdot 0038{,}4207)\\ &= \cdot 2774{,}2504.\end{aligned}$$

We may therefore take as the most probable value of the correlation coefficient in the series of u parent populations to three decimal places, $\rho = \cdot 277$.

Starting with this value of ρ we have now to find for the 30 values of r, given above, their probability integrals. Miss David's Probability Integral Table† for the Correlation Coefficient will enable such values to be determined. Meanwhile she has provided the values for $\rho = \cdot 277$. The probability integral table for this value of ρ runs as follows:

Thus, if the samples were all derived from populations having the same correlation we should only in less than 4 % of cases get a more improbable result than that observed. It is thus unlikely that the thirty races have the same correlation coefficient between Head Length and Head Breadth.

Messrs E. S. Pearson and S. S. Wilks, by applying the approximate method of Fisher, obtain a $\chi^2 = 96{\cdot}01$ and a probability $P_{\chi^2} < \cdot 000{,}030$ that the correlation

* *A priori* the hypothesis of common correlation is extremely unlikely, for if these races were produced by selection from a common stock, that selection would modify the correlations.

† Manuscript Table shortly to be published.

TABLE VII.

Ordinates and Probability Integrals of samples of 20 from a Parent Population of correlation $\rho = 0{\cdot}277$.

	Ordinates	Prob. Int. correct to 5 figs.	δ^2	δ^4
−1·00	—	—	—	—
−·95	—	—	—	—
−·90	—	—	—	—
−·85	—	—	—	—
−·80	·01	—	—	1
−·75	·03	—	1	−2
−·70	·13	·00001	—	4
−·65	·45⁺	·00002	3	—
−·60	1·27	·00006	6	6
−·55	3·17	·00016	15	3
−·50	7·10	·00041	27	12
−·45	14.55⁻	·00093	51	11
−·40	27·72	·001,96	86	17
−·35	49·59	·003,85⁻	138	23
−·30	83·92	·007,12	213	20
−·25	135·13	·012,52	308	24
−·20	207·94	·021,00	427	21
−·15	306·81	·033,75⁻	567	12
−·10	435·13	·052,17	719	− 3
−·05	594·15⁻	·077,78	868	− 19
·00	781·89	·112,07	998	− 43
·05	992·07	·156,34	1085	− 70
·10	1213·39	·211,46	1102	− 96
·15	1429·44	·277,60	1023	−105
·20	1619·53	·353,97	839	−117
·25	1760·79	·438,73	538	− 98
·30	1831·32	·528,87	139	− 60
·35	1814·42	·620,40	− 320	− 5
·40	1703·00	·708,73	− 784	67
·45	1503·12	·789,22	−1181	138
·50	1235·43	·857,90	−1440	184
·55	933·13	·912,18	−1515	195
·60	635·96	·951,31	−1395	162
·65	381·09	·976,49	−1113	76
·70	193·23	·990,54	− 755	− 22
·75	78·10	·997,04	− 419	− 91
·80	22·75	·999,35⁺	− 174	−120
·85	3·95	·999,92	− 49	− 84
·90	·33	1·000,00	− 8	− 33
·95	·00	1·000,00	—	− 8
1·0	·00	1·000,00	—	—

δ^6 is negligible.

Probability Integrals and their logarithms for observed values of r.

r_i	Prob. Int. correct to 4 figs.	Logarithm of P. Int.
+·097	·2085⁻	$\bar{1}$·319,1061
+·198	·3507	$\bar{1}$·544,9358
+·576	·9418	$\bar{1}$·973,9587
−·015	·1008	$\bar{1}$·003,4605
+·173	·3115⁺	$\bar{1}$·493,4581
+·764	·9980	$\bar{1}$·999,1305
−·037	·0858	$\bar{2}$·933,4873
+·667	·9823	$\bar{1}$·992,2441
+·014	·1234	$\bar{1}$·091,3152
−·112	·0472	$\bar{2}$·673,9420
+·219	·3853	$\bar{1}$·585,7990
−·152	·0331	$\bar{2}$·519,8280
+·319	·5637	$\bar{1}$·751,0480
+·310	·5473	$\bar{1}$·738,2254
+·019	·1277	$\bar{1}$·106,1909
+·445	·7816	$\bar{1}$·892,9846
+·410	·7256	$\bar{1}$·860,6973
+·946	1·0000⁻	·000,0000
+·018	·1268	$\bar{1}$·103,1193
+·160	·2921	$\bar{1}$·465,5316
+·178	·3100	$\bar{1}$·491,3617
+·763	·9979	$\bar{1}$·999,0870
+·101	·2127	$\bar{1}$·327,7675
+·449	·7877	$\bar{1}$·896,3608
+·245	·4300	$\bar{1}$·633,4685
+·360	·6385⁻	$\bar{1}$·805,1609
+·592	·9460	$\bar{1}$·975,8911
−·515	·0003	$\bar{4}$·477,1213
+·023	·1311	$\bar{1}$·117,6027
+·259	·4458	$\bar{1}$·649,1401

$\log \lambda_n = -17{\cdot}578,5760$

$\sqrt{30} \times {\cdot}434,2945 = 2{\cdot}378,7289^5$,

or $u = 7{\cdot}389,902$.

Accordingly

$$Q\lambda_n = 1 - I(29,\ 7{\cdot}389,902)$$
$$= 1 - {\cdot}963 = {\cdot}037.$$

coefficients between head length and head breadth are not the same for all the thirty races. I am not prepared to state whether the extreme difference in this case is due to the test applied being really more stringent, or to the fact that Fisher's approximate z-test for r can give exaggeratedly improbable values in the

case of outlying values of r such as those of $+\cdot946$ and $-\cdot592$ attributed to the Guanche and the Turk.

Illustration 6. *Use of the Incomplete* Γ*-Function Table to find the Probability Integrals.* We proceed first to indicate the Γ-function expression for the Probability Integral p_n of the Standard Deviation s of a random sample of n drawn from a normal curve of mean M and standard deviation Σ.

The equation for its distribution is

$$y = y_0 \left(\frac{s}{\Sigma/\sqrt{2n}}\right)^{n-2} e^{-\frac{1}{4}\left(\frac{s}{\Sigma/\sqrt{2n}}\right)^2} \quad [ds] \qquad \text{.....(26).}$$

Take $x = \frac{1}{4}\left(\frac{s}{\Sigma/\sqrt{2n}}\right)^2$, and we have

$$y = y_0' \, x^{\frac{n-3}{2}} e^{-x} \, dx \qquad \text{.....(27).}$$

Therefore the probability integral in the form ready for entering the incomplete Γ-function table is

$$p_n = I\left(\tfrac{1}{2}(n-3), \frac{ns^2}{\sqrt{2}(n-1)\Sigma^2}\right).$$

If we suppose the means and standard deviations are due to random sampling from a common normal population, then we have, by (12) and (13),

$$M = \frac{\overset{t=v}{S}(n_t m_t)}{N}, \quad \text{and} \quad \Sigma^2 = \frac{\overset{t=v}{S} n_t (s_t^2 + (m_t - M)^2)}{N}.$$

If we apply these results to the data in Table I we have $\Sigma^2 = 289{,}749$ and $n_t = 5$ for all values of t. Hence

$$p_{n_t} = I\left(1, \frac{s_t^2}{163907}\right).$$

Taking the values from column (f) of Table II we find

TABLE VIII.

t	$u = s_t^2/163{,}907$	p_{n_t}	$\log p_{n_t}$	t	$u = s_t^2/163{,}907$	p_{n_t}	$\log p_{n_t}$
1	1·18116	·4973	$\bar{1}$·696,6185	9	·90432	·3656	$\bar{1}$·563,0062
2	1·15447	·4853	$\bar{1}$·686,0103	10	1·29098	·5447	$\bar{1}$·736,1574
3	2·05238	·7855	$\bar{1}$·895,1462	11	1·91328	·7523	$\bar{1}$·876,3911
4	·72617	·2741	$\bar{1}$·437,9090	12	·47832	·1480	$\bar{1}$·170,2617
5	·51310	·1649	$\bar{1}$·217,2207	13	1·31919	·5562	$\bar{1}$·745,2310
6	1·58688	·6559	$\bar{1}$·816,8376	14	1·55591	·6452	$\bar{1}$·809,6944
7	1·20815	·5093	$\bar{1}$·706,9737	15	·88099	·3538	$\bar{1}$·560,8627
8	·60537	·2115	$\bar{1}$·325,3104	—	—	—	—

$S(\log p_{n_t}) = \log \lambda_n = -5{\cdot}756{,}3691, \quad \sqrt{n}\log_{10} e = 1{\cdot}682{,}0153, \quad u = 3{\cdot}42233.$

Thus: $I(14, 3{\cdot}42233) = {\cdot}3512$, or some $65\,^{\circ}/_{\circ}$ of series of 15 samples of 5 lamps would have a more improbable set of standard deviations.

Illustration 7. In *Illustration* 1 we have obtained the probability integrals of 15 tests of means and in the last illustration of 15 tests of standard deviations. As we have seen, probability integrals may be combined, and this is possible here because there is no correlation in samples from a normal population between mean and standard deviation.

The combined $\log \lambda_n = -8{\cdot}547{,}8434 - 5{\cdot}756{,}3691,$

$$= -14{\cdot}304{,}2125,$$

and $\sqrt{n} \log_{10} e = \sqrt{30} \times {\cdot}434{,}2945 = 2{\cdot}3787{,}2895.$

Accordingly: $u = 6{\cdot}01338$, and

$$I(n-1, u) = I(29, 6{\cdot}01338) = {\cdot}7189.$$

and $1 - I(29, 6{\cdot}01338) = {\cdot}281$, or there would be samples of 15 drawn from a normal surface some 28 % with more improbable sets of means and standard deviations than occur in this set of 15 samples of five lamps. Thus from whatever standpoint we regard the problem, we have not succeeded in condemning the hypothesis that the 15 samples of five lamps each may have been drawn from the *same* normal population.

But there is a considerable difference in the combined z-test and the combined λ_n-test (i.e. $Q_{\lambda_{15}} = {\cdot}070$ and $Q_{\lambda_{30}} = {\cdot}281$). We have already pointed out that the z-test only involves the populations from which the individual z's are obtained having the same mean and being normal; these populations may have *different* standard deviations, and if the selections from each of these distinct populations be random, their probability integrals for z will all follow a rectangular distribution and may be combined. On the other hand our combined test assumes normal parent populations with the same mean and the *same* standard deviation. We should thus expect it to be more stringent than the z-test; actually it is less. But this may be accounted for by another factor which arises here: the z-test associates every difference of mean with a definite standard deviation.

In the present illustration the 15 standard deviations, s_i, of the samples might have been associated with any one of the 15 differences of mean, $m_i - M$. In this respect the combined test of this illustration seems to be less stringent than the z-test of *Illustration* 2.

Accordingly we ought to be very careful in considering a test to state the hypothesis we are testing on as wide a basis as it admits in regard to the methods employed.

Thus in *Illustration* 2 we are not really testing whether the 15 means come from a single normal population with the same mean and standard deviation. We are testing whether the 15 means come from 15 normal populations with the same mean and possibly different standard deviations. If the result were such that we rejected the latter hypothesis, the former must be rejected for it is involved in the latter. But if the latter be reasonable, it does not follow that the former also may be.

In *Illustrations* 1 and 5 we have made—the hypothesis that all the 15 samples were drawn from the same normal population, and calculated its mean and standard deviation as the most probable values on the basis of the data provided by the 15 samples. But we have not specified that the 15 standard deviations are to be associated with special values of the differences of the means. Had the result come out highly improbable we should have rejected the wider hypothesis, and accordingly the narrower, which is included in it.

One point may be noted. The probability of a worse result for the means $= \cdot 6488$ and for a worse result for the standard deviations $= \cdot 1179$. The combined improbability of a worse result for both, as these are independent, $= \cdot 0765$, which is of the same order of probability as the $\cdot 0696$ provided by the z-test in *Illustration* 2. The correspondence is possibly of no significance.

If we take the 30 probability integrals provided by *Illustrations* 1 and 5, and distribute them in five subranges of $\cdot 2$, we find

	0·0—0·2	0·2—0·4	0·4—0·6	0·6—0·8	0·8—1·0
Expected	6	6	6	6	6
Observed	6	6	7	8	3

The $\chi^2 = 2 \cdot 3333$, and the corresponding P_{χ^2} for five groups $= \cdot 676$.

This may be compared with the $P_{\lambda_n} = \cdot 281$, as indicating the weakness of the assumptions on which the χ^2, P method depends for a small number of groups and a small total frequency.

Illustration 8. *Application to Linearity of Regressions.* In some recent work by Professor H. Ruger the correlation tables for Weight and Vital Capacity in 28 age groups were reached and the values of r, $\eta_{x.y}$, $\eta_{y.x}$ calculated for each of these groups. From these values the functions $\zeta_{x.y} = (\eta^2_{x.y} - r^2)/(1 - r^2)$ and $\zeta_{y.x} = (\eta^2_{y.x} - r^2)/(1 - r^2)$ were tabled. The distribution of ζ on the assumptions made by Fisher of linearity of regression, with homoscedastic normal arrays, leads to the curve

$$y = y_0 \zeta^{\frac{1}{2}(a-4)} (1 - \zeta)^{\frac{1}{2}(n-a-2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

where n is the size of the sample and a the number of arrays; and accordingly the probability integral of ζ could be found from the *Tables of the Incomplete* B-*Function* as long as $n - a - 2 \leqslant 100$.

Unfortunately in the data we have referred to this condition is only satisfied in one case; in the others $N - a - 2$ is in excess and often considerably in excess of 100. We therefore replace in (28) the second factor by an exponential term, and we have, if $\xi = \frac{1}{2}(n - a - 2)\zeta$, the distribution

$$y = y_0' \cdot \xi^{\frac{1}{2}(a-4)} e^{-\xi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

The probability integral will accordingly be $I(\frac{1}{2}(a-4), u)$, where

$$u = \frac{\frac{1}{2}(n-a-2)}{\sqrt{\frac{1}{2}(a-2)}}\zeta,$$

and $I(p, u)$ is the incomplete Γ-function ratio to be sought for in the *Tables of the Incomplete* Γ-*Function*. This is feasible up to 104 arrays, a number unlikely to be required.

The reader must bear in mind that a is the number of arrays on which η is computed, and we must distinguish $a_{x.y}$ corresponding to $\eta_{x.y}$ from $a_{y.x}$ corresponding to $\eta_{y.x}$.

The problem before us is the following: we have 28 tables and 28 values of r and say $\eta_{y.x}$, i.e. Weight on Vital Capacity. May the 28 regression lines of Weight on Capacity be considered as a system, which does not differ from straight lines

TABLE IX.

Regression of Weight on Vital Capacity.

No. of Sample t	Age Group	Size n_t	$a_{y.x}$	$\frac{1}{2}(a_{y.x}-4)$	$n_t - a_{y.x} - 2$	$\zeta_{y.x}$	$\sqrt{2(a_{x.y}-2)}$	u	p_t	$\log p_t$
1	6—12	105	13	$\begin{cases} p=5{\cdot}5 \\ q=46 \end{cases}$	—	·247,5080	—	—	·004	$\bar{3}$·602,0600
2	13—15	331	15	5·5	314	·067,7070	5·099,0195	4·16943	·068	$\bar{2}$·832,5089
3	16	241	15	5·5	224	·068,6530	5·099,0195	3·01593	·285	$\bar{1}$·454,8449
4	17	288	14	5	272	·031,4678	4·898,9795	1·74715	·022	$\bar{2}$·342,4227
5	18	320	15	5·5	303	·040,2568	5·099,0195	2·39219	·512	$\bar{1}$·709,2700
6	19	310	18	7	290	·055,5070	5·656,8542	2·84558	·446	$\bar{1}$·649,3349
7	20	389	17	6·5	370	·013,1104	5·477,2256	0·88564	·993	$\bar{1}$·996,9492
8	21	321	16	6	303	·057,4679	5·291,5026	$3{\cdot}29070^5$	·235	$\bar{1}$·371,0679
9	22	289	16	6	271	·009,2677	5·291,5026	0·47464	·999(64)	$\bar{1}$·999,8436
10	23	316	17	6·5	297	·063,0081	5·477,2256	3·41658	·227	$\bar{1}$·356,0259
11	24	276	14	5	260	·067,7565	4·898,9795	3·59599	·128	$\bar{1}$·107,2100
12	25	223	17	6·5	204	·063,3770	5·477,2256	2·36048	·608	$\bar{1}$·783,9036
13	26	224	16	6	226	·049,5475	5·291,5026	2·11617	·670	$\bar{1}$·826,0748
14	27	195	15	5·5	178	·075,0919	5·099,0195	2·62136	·430	$\bar{1}$·633,4685
15	28	188	16	6	170	·085,6810	5·291,5026	2·75267	·409	$\bar{1}$·611,7233
16	29	186	15	5·5	169	·096,6815	5·099,0195	3·20438	·231	$\bar{1}$·363,6120
17	30	198	14	5	182	·068,2614	4·898,9795	2·53595	$·412^5$	$\bar{1}$·615,4240
18	31—32	305	17	6·5	286	·081,0745	5·477,2256	4·23366	·080	$\bar{2}$·903,0900
19	33—34	280	16	6	262	·036,9507	5·291,5026	1·82955	·785	$\bar{1}$·894,8697
20	35—36	290	16	6	272	·056,4152	5·291,5026	2·89992	·355	$\bar{1}$·550,2284
21	37—38	266	16	6	248	·036,2694	5·291,5026	1·69986	·831	$\bar{1}$·919,6010
22	39—41	319	16	6	301	·040,0175	5·291,5026	2·26878	·606	$\bar{1}$·782,4726
23	42—44	267	14	5	251	·022,8722	4·898,9795	1·17186	·282	$\bar{1}$·450,2491
24	45—47	196	16	6	178	·055,8809	5·291,5026	1·87977	·766	$\bar{1}$·884,2288
25	48—51	222	16	6	204	·023,4922	5·291,5026	0·90568	·884	$\bar{1}$·946,4523
26	52—55	147	15	5·5	130	·097,3283	5·099,0195	2·48139	·475	$\bar{1}$·676,6936
27	56—61	186	14	5	170	·055,0076	4·898,9795	1·90882	·673	$\bar{1}$·828,0151
28	62—81	163	17	6·5	144	·150,1917	5·477,2256	3·94864	·118	$\bar{1}$·071,8820

p_t gives the probability of a higher $\zeta_{y.x}$ occurring than that observed.

g$_1$ $\lambda_n = -14{\cdot}836,4732$, $\sqrt{n}\log_{10}e = 2{\cdot}298,0705$. Hence $u = 6{\cdot}45606$.

more than would reasonably be the result of random sampling? In other words are the 28 ξ's as based on the 28 ζ's a random sample from the distribution curve (29)?

For the tth sample, we require n_t the size of the sample, $a_{y.x}$, the number of arrays of Weight on Vital Capacity, the $\zeta_{y.x}$, $\sqrt{2(a_{y.x}-4)}$, $n_t - a_{y.x} - 2$ and the $p = \frac{1}{2}(a_{y.x}-4)$ of the incomplete Γ-function, and lastly the value of

$$u = \frac{n_t - a_{y.x} - 2}{\sqrt{2(a_{y.x}-4)}}\zeta_{y.x}.$$

Hence from p and u we find the probability integral p_t of the sample and record its logarithm. Their sum gives the required $\log_{10}\lambda_n$.

Thus we find $I(27,\ 6{\cdot}45606) = {\cdot}8748$ and sets of 28 tables between Weight and Vital Capacity for age groups like the above, if obtained as random samples from parent populations with a linear regression of Weight on Vital Capacity would give a less probable set in 12·5 % of cases. Our results therefore taken as a whole do not provide strong evidence of non-linear regression in the case of Weight on Vital Capacity.

Illustration 9. *Random Samples of Correlation Coefficient from a Normal Population of zero Coefficient.* The following system of racial correlation coefficients for Cephalic Index (100 B/L) and Upper Face Index is given by Tippett*. Applying an approximate method he concludes as follows:

"Thus the combined experience of Table XLIII [this corresponds to our Table X below] lends no support to the view that the two characters are associated even after making allowance for the possibility of racial differences," p. 143.

It seems worth while investigating whether the methods of the present paper confirm Tippett's conclusions.

TABLE X.

Correlation Coefficients of Cephalic Index with Upper Face Index for thirteen Races†.

Index No.	Race	Size of Sample n_t	Correlation Coefficient r_t	Index No.	Race	Size of Sample n_t	Correlation Coefficient r_t
1	Australians ...	66	+0·089	7	Polynesians ...	44	+0·002
2	Negroes ...	77	+0·182	8	Alfourons ...	19	−0·302
3	Duke of York Islanders	53	−0·093	9	Micronesians ...	32	−0·251
				10	Copts	34	−0·147
4	Malays ...	60	−0·185	11	Etruscans ...	47	−0·021
5	Fijians ...	32	+0·217	12	Europeans ...	80	−0·198
6	Papuans ...	39	−0·255	13	Ancient Thebans	152	−0·067

* *The Methods of Statistics*, p. 142, 1931.

† Several of the groups are scarcely anthropological unities, but the series will serve as an example of method.

We will first find the most probable value of ρ, the correlation of the parent populations, if they all had the same coefficient. Applying the method of p. 394 above we find, if $N = S(n_t)$,

$$\frac{S(n_t r_t)}{N} = -\cdot 060{,}9878, \quad \frac{S(x_t r_t^2)}{N} = \cdot 024{,}4213,$$

$$\frac{S(n_t r_t^3)}{N} = -\cdot 002{,}7694, \quad \frac{S(n_t r_t^4)}{N} = \cdot 001{,}1779.$$

Hence we deduce

$$\rho_1 = -\cdot 060{,}9878, \quad \rho_2 = -\cdot 062{,}2504, \quad \rho_3 = -\cdot 062{,}2763, \text{ and } \quad \rho_4 = -\cdot 062{,}2772.$$

Accordingly we find as the most probable value for ρ

$$\rho = -\cdot 06228.$$

It cannot therefore be asserted that the most probable value of ρ as indicated by the data is zero. It looks as if there existed a small negative correlation between the First Cephalic and the Upper Face Indices.

Let us first try what the series leads to when we replace this most probable value of ρ by $\rho = 0$. In this case the frequency distribution of r as a selection from a population having zero correlation coefficient, $\rho = 0$, is given by

$$y = y_0 (1 - r^2)^{\frac{n-4}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

and accordingly the probability integral is given by

$$p_{n,\,t} = \int_{-1}^{r} (1 - r^2)^{\frac{1}{2}(n-4)}\, dr \Big/ \int_{-1}^{+1} (1 - r^2)^{\frac{1}{2}(n-4)}\, dr \quad \ldots\ldots\ldots\ldots (31).$$

This can be reduced to the incomplete B-function ratio by one or other of two transformations.

(i) Take $r^2 = x$, and we find

$$p_{n,\,r} = 1 - \tfrac{1}{2} I_{1-r^2}(\tfrac{1}{2}(n-2), \tfrac{1}{2}), \text{ if } r \text{ be positive,}$$
$$= \tfrac{1}{2} I_{1-r^2}(\tfrac{1}{2}(n-2), \tfrac{1}{2}), \text{ if } r \text{ be negative.}$$

(ii) Put $\tfrac{1}{2}(1 + r) = x$, and we reach

$$p_{n,\,r} = I_{\frac{1}{2}(1+r)}(\tfrac{1}{2}(n-2), \tfrac{1}{2}(n-2)), \text{ for both signs of } r.$$

We shall make use of the second transformation as more readily lending itself to interpolation* into the *Incomplete* B-*Function Tables.*

* I have considered it adequate for present purposes to interpolate linearly into the *Tables of the Incomplete* B-*Function*, but even with this simplification the determination of $I_{x+\theta d}(p+\frac{1}{2}, p+\frac{1}{2})$ causes some little trouble, when we are using that part of the tables wherein p is only given to the unit; further we have to remember that $I_x(p, q)$ is only tabled for $p > q$, so that we must find when q is $> p$, $I_x(p, q)$ from the relation $I_x(p, q) = 1 - I_{1-x}(q, p)$. The requisite formula is the following, where $\phi = 1 - \theta$ and d, the tabulating interval of x, is here unity:

$$I_{x+\theta}(p+\tfrac{1}{2}, p+\tfrac{1}{2}) = \tfrac{1}{4}\phi\{I_x(p, p) + 1 - I_{1-x}(p+1, p) + I_x(p+1, p+1) + I_x(p+1, p)\}$$
$$+ \tfrac{1}{4}\theta\{I_{x+d}(p, p) + 1 - I_{1-x-d}(p+1, p) + I_{x+d}(p+1, p+1) + I_{x+d}(p+1, p)\}.$$

Here all the incomplete B-function ratios will be found in the tables, as p is an integer.

TABLE XI.

P_λ Test for Tippett's Data.

Index Number t	Size of Sample n_t	Correlation r_t	Value of p_{n_t, r_t}		Logarithm of p_{n_t, r_t}
			β-function Value	Numerical Value	
1	66	+·089	$I_{\cdot 5443}(32, 32)$	·761	$\bar{1}$·881,3847
2	77	+·182	$I_{\cdot 5910}(37{\cdot}5, 37{\cdot}5)$	·946	$\bar{1}$·975,8911
3	53	−·093	$I_{\cdot 4535}(25{\cdot}5, 25{\cdot}5)$	·255	$\bar{1}$·406,5402
4	60	−·185	$I_{\cdot 4075}(29, 29)$	·079	$\bar{2}$·897,6271
5	32	+·217	$I_{\cdot 6085}(15, 15)$	·117	$\bar{1}$·068,1859
6	39	−·255	$I_{\cdot 3725}(18{\cdot}5, 18{\cdot}5)$	·060	$\bar{1}$·778,1513
7	44	+·002	$I_{\cdot 5010}(21, 21)$	·505	$\bar{1}$·703,2914
8	19	−·302	$I_{\cdot 3490}(8{\cdot}5, 8{\cdot}5)$	$\cdot 104^5$	$\bar{1}$·019,1163
9	32	−·251	$I_{\cdot 3745}(15, 15)$	·083	$\bar{2}$·919,0781
10	34	−·147	$I_{\cdot 4265}(16, 16)$	·203	$\bar{1}$·307,4960
11	47	−·021	$I_{\cdot 4895}(22{\cdot}5, 22{\cdot}5)$	·445	$\bar{1}$·648,3600
12	80	−·198	$I_{\cdot 4010}(39, 39)$	·039	$\bar{2}$·591,0646
13	152	−·067	$I_{\cdot 4665}(75, 75)$*	$\cdot 207^5$	$\bar{1}$·317,0181

$\log \lambda_n = \text{sum } \log p_{n_t, r_t} = -8{\cdot}486,7952$, $\sqrt{13}\log_{10} e = 1{\cdot}565,8711$, $u = 5{\cdot}41986$.

Chance of more improbable sets $= 1 - I(12, 5{\cdot}41986) = {\cdot}048$.

Tippett applying a normal curve of standard deviation $\frac{1}{\sqrt{n-3}}$ instead of (30) finds a χ^2 for 13 lying between 17 and 18 (17·26) which leads to a $P = {\cdot}141$, or between the ·1 and ·2 levels. It would thus appear that our test is more stringent than that applied by Tippett, or the difference may be due to the approximate nature of the method used by him. Not being a very enthusiastic advocate of ·02 as a fit measure for rejection of randomness, I am inclined to doubt whether ·048 is to be taken as sufficient evidence that the series of correlation coefficients are random samples of populations with zero coefficients of correlation.

Illustration 10. *Comparison of two Hypotheses.* We have already noted that, apart from the replacement of binomials by normal curves, the χ^2 test suffers under the disadvantage that it gives the same resulting probability wherever the individuals may be in the same set of subranges. The $(\lambda_n, P_{\lambda_n})$ test allows for this, and accordingly is far better suited for answering the problem of whether a Hypothesis *A* is or is not more probable than a Hypothesis *B*.

We will illustrate such a comparison by Tippett's data in *Illustration* 9. In that illustration we have taken Tippett's hypothesis that the data are random samples from parent populations having zero correlations. This shall be Hypothesis *A*. We have seen that the most likely value of the corre'ation is not zero, but a correlation measured by −·06228. We will ask what is the probability that the thirteen samples were drawn from parent populations with the correlation of the variates measured by the coefficient −·06228. This is our Hypothesis *B*.

* Obtained approximately from corresponding normal curve.

To answer the problem requires us to determine the probability integrals for $\rho = -\cdot 06228$ of thirteen samples ranging from size 19 to size 152. The tables of the probability integral of r as sampled from normal distributions are not yet sufficiently advanced to cover this wide field. Only the sample of 19 falls within the present range of those tables. It seemed best to adopt a uniform process for all cases, and accordingly the following method was used. It is known that from about $n = 20$ onwards an excellent fit to the distribution curve of r is obtained by aid of a Pearson curve* having the same first four moment coefficients† as the r-distribution. The resulting curves belong to Type I:

$$y = y_0 (a_1 + x)^{m_1' - 1} (a_2 - x)^{m_2' - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32),$$

and accordingly the probability integrals may all be found from the *Tables of the Incomplete* B-*Function*. In order to reduce these curves we need to find: (i) $m_1' < m_2'$, these are found from the β_1 and β_2; (ii) the range b, this is found from μ_2, β_1 and β_2 and (iii) the distance d of the observed correlation coefficient from the start of the Pearson curve. This must be reduced to d' by dividing by the range b, for entry into the B-function table.

In determining the incomplete B-function ratio we must remember that as m_2' is always greater than m_1' we must look up in the tables not $I_{d'}(m_1', m_2')$ but its equivalent $1 - I_{1-d'}(m_2', m_1') = p_n$. The probability integral thus obtained is the probability of the occurrence of samples with a greater improbability than the observed sample. Table XII gives the values of the constants of the curves, the corresponding p_n's and their logarithms and finally the value of u with which we enter the *Incomplete* Γ-*Function Table*. This gives by linear interpolation:

$$P_{\lambda_n} = 1 - I(12,\ 3{\cdot}674{,}963) = 1 - {\cdot}5640 = {\cdot}4360.$$

Thus by the hypothesis that Cephalic Index and Upper Face Index have a correlation coefficient equal to the most probable value provided by the whole set of experiences we obtain a probability rather more than nine times as great as that provided by the hypothesis that the correlation coefficient is really zero. To those who have had experience of the correlation between cranial characters in *long* series, the fact that small correlations between such may be significant is familar. There is accordingly no ground for assuming that because we have four positive and nine negative small correlation coefficients it is a reasonable hypothesis that the correlation coefficient between these two characters is zero.

But here I reach my main criticism of the method now frequently adopted for testing hypotheses. Arbitrary values like $P = \cdot 01$ or $P = \cdot 02$ are taken to indicate the improbability of a hypothesis. A hypothesis is then found to have a $P = \cdot 04$ or $\cdot 06$, and it is stated to be thus shown to be reasonable. A conclusion is then drawn from the hypothesis, which is taken as a physical principle, for examples that no difference exists between two populations, or that a correlation coefficient is zero. No attention is paid to the fact that another hypothesis may prove more

* *Biometrika*, Vol. XI. pp. 332—336, or *Tables for Statisticians*, Part II. pp. clvii—clxi.

† *Biometrika*, Vol. XI. pp. 337—338, or *Tables for Statisticians*, Part II. p. clxii.

TABLE XII. *Probability Integrals for Tippett's Series on the Hypothesis that* $\rho = -\cdot06228$.

Index Number and Size of Sample	Observed r	Mean r for Size of Sample $=\bar{r}$	Variance μ_2 for Size of Sample	β_1 for Size of Sample	β_2 for Size of Sample	Range of fitted Type I Curve $=b$	Value of r at Mode $=\bar{r}$	Distance of Mode from Start $=a_1$	Distance of Start from Observed r $=d$ (d')	m_1' m_2'	Probability Integral $1-I_{1-d'}(m_2', m_1')$ $=p_n$	$\log p_n$
(8) 19	−·302	−·0605,8054	·0551,8370	·006,004	2·710,552	2·0078,3827	−·0717,8237	·9079,5440	·6777,3677	7·843,568	·161	$\bar{1}$·206,8259
									(·3375,455)	9·290,208		
(5) 32	+·217	−·0612,8732	·0320,2889	·003,854	2·825,322	2·0098,9723	−·0676,2820	−·9089,4567	1·1935,7387	13·789,469	·936	$\bar{1}$·971,2758
									(·5938,482)	16·491,116		
(9) 32	−·251	−·0612,8732	·0320,2889	·003,854	2·825,322	2·0098,9723	−·0676,2820	−·9089,4567	·7255,7387	13·789,469	·150	$\bar{1}$·176,0913
									(·3609,995)	16·491,116		
(10) 34	−·147	−·0613,4702	·0300,8665	·003,654	2·835,371	2·0100,5458	−·0672,8157	−·9091,7394	·8294,5551	14·706,676	·319	$\bar{1}$·503,7907
									(·4126,532)	17·596,840		
(6) 39	−·255	−.0614,6897	·0261,2591	·003,234	2·856,071	2·0103,5996	−·0665,8482	·9096,1436	·7211,9918	16·999,381	·145	$\bar{1}$·161,3680
									(·3587,413)	20·361,225		
(7) 44	+·002	−·0615,6273	·0230,8665	·002,900	2·872,149	2·0106,1194	−·0660,5880	·9099,4799	·9780,0679	19·292,236	·662	$\bar{1}$·820,8580
									(·4864,225)	23·126,105		
(11) 47	−·021	−·0616,0925	·0215,8037	·002,731	2·880,179	2·0107,2473	−·0658,0109	·9100,9783	·9548,9892	20·667,489	·610	$\bar{1}$·785,5298
									(·4749,029)	24·784,880		
(3) 53	−·093	−·0616,8628	·0190,8939	·002,445	2·893,553	2·0109,4018	−·0653,7809	·9103,6524	·8827,4333	23·418,940	·515	$\bar{1}$·711,8072
									(·4389,705)	28·103,103		
(4) 60	−·185	−·0617,5642	·0168,2379	·002,179	2·906,005	2·0111,4740	−·0649,9769	·9105,9828	·7905,9597	26·628,856	·175	$\bar{1}$·243,0380
									(·3931,069)	31·975,036		
(1) 66	+·089	−·0618,0457	·0152,7035	·001,992	2·914,286	2·0113,0239	−·0647,3855	·9107,7362	1·0645,1217	29·380,999	·886	$\bar{1}$·947,4337
									(·5292,651)	35·294,039		
(2) 77	+·182	−·0618,7316	·0130,5958	·001,723	2·926,410	2·0114,7477	−·0643,7317	·9109,8490	1·1573,5807	34·424,315	·982	$\bar{1}$·992,1115
									(·5753,779)	41·377,311		
(12) 80	−·198	−·0618,8856	·0125,6352	·001,661	2·929,144	2·0115,0943	−·0642,9167	·9110,2893	·7673,2060	35·799,574	·097	$\bar{2}$·986,7717
									(·3814,651)	43·036,264		
(13)152*	−·067	−·0620,9490	·0065,7216	·000,895	2·962,547	2·0121,6079	−·0633,3995	·9250,9024	·9214,3019	68·820,716	·548	$\bar{1}$·738,7806
									(·4579,307)	82·871,001		

$\sqrt{n}\log_{10}e = 1\cdot565,8711$, $\quad u = 3\cdot674,963$. $\qquad S(\log p_n) = \log \lambda_n = -5\cdot754,5178$.

* The frequency is too great to bring this case into the range of the *Incomplete* B-*Function Table*, and accordingly the normal curve was used.

stringent* and indicate a difference between two populations, or show that it is more reasonable to suppose the correlation coefficient not zero.

Illustration 9. *Comparison of two Hypotheses.* I will take a further illustration of such a comparison which will also cast more light on the difficulties I feel with regard to small samples.

The following thirty observations given in column (*b*) of Table XIII are a random sample and we take as our Hypothesis *A* that the parent population had

TABLE XIII.

Test for Normality in Parent Population from Sample.

(*a*) Index No.	(*b*) Observation x_i	(*c*) Deviation from Mean x_i'	(*d*) x_i'/Σ	(*e*) p_i†	(*f*) $\log_{10} p_i$
1	25	−419·5	−1·485	·0688	$\bar{2}$·837,5884
2	550	+105·5	+ ·373	·6454	$\bar{1}$·809,8290
3	517	+ 72·5	+ ·257	·6014	$\bar{1}$·779,1634
4	33	−411·5	−1·457	·0726	$\bar{2}$·860,3380
5	210	−234·5	− ·830	·2033	$\bar{1}$·308,1374
6	641	+196·5	+ ·696	·7568	$\bar{1}$·878,9811
7	477	+ 32·5	+ ·115	·5458	$\bar{1}$·737,0335
8	318	−126·5	− ·448	·3271	$\bar{1}$·514,6805
9	418	− 26·5	− ·094	·4626	$\bar{1}$·665,2056
10	277	−167·5	− ·593	·3766	$\bar{1}$·575,8803
11	532	+ 87·5	+ ·310	·6217	$\bar{1}$·793,5809
12	466	+ 21·5	+ ·076	·5303	$\bar{1}$·724,5216
13	671	+226·5	+ ·802	·7887	$\bar{1}$·896,9118
14	595	+150·5	+ ·533	·7030	$\bar{1}$·846,9553
15	152	−292·5	−1·035	·1503	$\bar{1}$·176,9590
16	988	+543·5	+1·924	·9728	$\bar{1}$·988,0236
17	420	− 24·5	− ·087	·4653	$\bar{1}$·667,7331
18	625	+180·5	+ ·639	·7386	$\bar{1}$·868,4093
19	389	− 55·5	− ·196	·4333	$\bar{1}$·636,7887
20	171	−273·5	− ·968	·1665	$\bar{1}$·221,4142
21	968	+523·5	+1·853	·9681	$\bar{1}$·985,9202
22	728	+283·5	+1·004	·8423	$\bar{1}$·925,4668
23	949	+504·5	+1·786	·9629	$\bar{1}$·983,5812
24	178	−266·5	− ·943	·1728	$\bar{1}$·237,5437
25	120	−324·5	−1·149	·1253	$\bar{1}$·097,9511
26	144	−300·5	−1·064	·1437	$\bar{1}$·157,4568
27	37	−407·5	−1·443	·0745	$\bar{2}$·872,1563
28	944	+499·5	+1·768	·9615	$\bar{1}$·982,9493
29	289	−155·5	− ·550	·2912	$\bar{1}$·464,1914
30	503	+ 58·5	+ ·207	·5820	$\bar{1}$·764,9230
Mean = 444·5		Σ = 282·49328		$\log_{10}\lambda_n$ = −12·739,7252	

$$\frac{1}{\sqrt{30}\log_{10} e} = \cdot 4203,9259,67, \quad I(n-1, u) = I(29, 5\cdot 3556,8616) = \cdot 47551.$$

* For illustrations in degrees of stringency of different tests, see *Biometrika*, Vol. XXIV. pp. 305 *et seq.*

† Found by linear interpolation from Sheppard's Tables.

a normal distribution. The mean M of the observations is $M = 444{\cdot}5$ and their standard deviation $\Sigma = 282{\cdot}49328$. We will adopt these as probable values in the parent population. Column (c) of Table XIII gives the deviations from the mean; column (d) expresses these in terms of the standard deviation, column (e) gives the probability integrals p_t, and column (f) their logarithms the sum of which is $-12{\cdot}739{,}7252$. Accordingly since $1/\sqrt{n}\log_{10}e = {\cdot}4203{,}9259{,}67$, we require to find $I(29, 5{\cdot}3556{,}8616)$ from the *Table of the Incomplete* Γ-*Function*. Interpolating by means of δ^2 (i.e. to third difference accuracy), we have

$$P_{\lambda_n} = I(29,\ 5{\cdot}3556{,}8616) = {\cdot}47551, \quad \text{and} \quad Q_{\lambda_n} = {\cdot}52449.$$

Thus between 52 °/₀ and 53 °/₀ of samples from the above mentioned normal population would have a less degree of probability than the observed sample. Shall we therefore assume it "reasonable" to suppose the sample drawn from a normal population? I fear many statisticians will say that it is, and not hesitate to draw any inferences that may be based on such an assumption. I doubt, however, whether the use of the word "reasonable" is proper in a case of this kind. There are possibly far more probable hypotheses as to the nature of the parent population, which might lead us to base very different conclusions on the nature of the sample, for example that the range of possible observations was narrowly limited, or that the frequency of the parent population was not such that small or large values of the observations occurred with relatively small frequency. We will take a second hypothesis, Hypothesis B, that the sample has been drawn from a rectangular population. The maximum range as shown by the sample $= 988 - 25 = 963$. This is most probably the modal range of samples from the parent population $= (n-2)\,b/(n-1)$* $= \frac{28}{29}\,b$, where b is the range of the parent population. Hence $b = \frac{29}{28} \times 963 = 997{\cdot}39286$. This seems a good value to take for the range of the parent population. Table XIV, column (b) gives the observations; (c) the probability integrals; (d) their logarithms, with their sum, leading to the incomplete Γ-function $P_{\lambda_n} = I(29, 6{\cdot}2168{,}5177)$, the evaluation of which by δ^2 interpolation from the *Incomplete* Γ-*Function Tables* is equal to ·77905, and $Q_{\lambda_n} = {\cdot}22095$.

Accordingly almost 22 °/₀ of samples of sets of observations from a rectangular parent of the above range would be more improbable than the observed set, and the Hypothesis A is seen to be much more probable than Hypothesis B, although if Hypothesis B had been first tried and its probability, ·22095, computed, many statisticians would have been content with its "reasonableness," and not have proceeded further. Now the strange fact is that the observations were actually taken as the first three figures of the first six sets of five on sheet XXIV of Tippett's *Random Sampling Numbers*, and may therefore be supposed to form a random sample of 30 from a rectangular parent population of range 1000. Our new method should enable us fairly readily to compare the probability of different hypotheses. But the main point to be noted is that because one hypothesis has a

* *Biometrika*, Vol. XXIII. p. 394.

TABLE XIV.

Test for Rectangularity of Parent Population from Sample.

(*a*) Index No.	(*b*) Observation	(*c*) $p_t=(b)\div 997{\cdot}39286$	(*d*) $\log_{10} p_t$	(*a*) Index No.	(*b*) Observation	(*c*) $p_t=(b)\div 997{\cdot}39286$	(*d*) $\log_{10} p_t$
1	25	·0251	$\bar{2}{\cdot}399{,}6737$	16	988	·9906	$\bar{1}{\cdot}995{,}8983$
2	550	·5514	$\bar{1}{\cdot}741{,}4668$	17	420	·4211	$\bar{1}{\cdot}624{,}3852$
3	517	·5184	$\bar{1}{\cdot}714{,}6650$	18	625	·6266	$\bar{1}{\cdot}796{,}9904$
4	33	·0331	$\bar{2}{\cdot}519{,}8280$	19	389	·3900	$\bar{1}{\cdot}591{,}0646$
5	210	·2105	$\bar{1}{\cdot}323{,}2521$	20	171	·1714	$\bar{1}{\cdot}234{,}0108$
6	641	·6427	$\bar{1}{\cdot}808{,}0083$	21	968	·9705	$\bar{1}{\cdot}986{,}9955$
7	477	·4782	$\bar{1}{\cdot}679{,}6096$	22	728	·7299	$\bar{1}{\cdot}863{,}2634$
8	318	·3188	$\bar{1}{\cdot}503{,}5183$	23	949.	·9515	$\bar{1}{\cdot}978{,}4088$
9	418	·4191	$\bar{1}{\cdot}622{,}3177$	24	178	·1785	$\bar{1}{\cdot}251{,}6382$
10	277	·2777	$\bar{1}{\cdot}443{,}5759$	25	120	·1203	$\bar{1}{\cdot}080{,}2656$
11	532	·5334	$\bar{1}{\cdot}727{,}0530$	26	144	·1444	$\bar{1}{\cdot}159{,}5672$
12	466	·4672	$\bar{1}{\cdot}669{,}5028$	27	37	·0371	$\bar{2}{\cdot}569{,}3739$
13	671	·6728	$\bar{1}{\cdot}827{,}8860$	28	944	·9465	$\bar{1}{\cdot}976{,}1206$
14	595	·5966	$\bar{1}{\cdot}775{,}6832$	29	289	·2898	$\bar{1}{\cdot}462{,}0984$
15	152	·1524	$\bar{1}{\cdot}182{,}9850$	30	503	·5043	$\bar{1}{\cdot}702{,}6890$
p_t values retained only to four places						$\log_{10}\lambda_n=-14{\cdot}788{,}2047$	
$P_{\lambda_n}=I(n-1,\ 6{\cdot}2168{,}5177)={\cdot}77905$*						$\frac{-\log_{10}\lambda_n}{\sqrt{n}\,\log_{10}e}=6{\cdot}2168{,}5177$	

very considerable probability and a higher probability than a second, it does not follow that it is reasonable to suppose that hypothesis to hold and another to be false, and hence draw conclusions from the former holding†.

The effect of using *small* samples is to render it quite probable (probability = ·50 about!) that a sample was drawn from a population differing very widely from the population out of which it was actually extracted. In short the fact that a hypothesis is even very probable in the case of a small sample by no means demonstrates that it is a "reasonable" hypothesis, and that accordingly inferences may be drawn from it‡. All in fact that the present test and other tests besides

* If b be found from $\Sigma^2=\frac{1}{12}b^2$, it equals 978·592,356 and $P_{\lambda_n}=$·75012. But this b is less than the observation 988 and gives one probability value =1·0096, i.e. >1!

† The greater probability of the normal hypothesis is here explicable because we have *two* constants to dispose of, while in the case of the rectangle, we have only made use of the range, measuring it from the zero of observation. It is not very easy, short of approximations of considerable length, to determine the best "centre" and the best range of a rectangular population from a given sample, and clearly any attempt to do so moves us away from the true parent population.

‡ I have elsewhere (*Biometrika*, Vol. XXIV. p. 371) indicated that given two parent populations as divergent in distribution as the normal and rectangular, it is not possible to deny the truth of one or other hypothesis unless the sample approaches 100 to 150 in magnitude.

can achieve if several hypotheses are found to have considerable probability is to test their *relative* reasonableness, and even this may deceive us, as just exemplified.

Conclusions.

(i) A very general test, the P_{λ_n} test, has been discussed which seems to the writer to involve fewer approximations and assumptions than the P_{χ^2} test. He would emphasise its advantages in this respect in the case of small samples, where it appears to him that the application of the P_{χ^2} test may well lead to erroneous conclusions, for it fails in stringency.

(ii) The P_{λ_n} is not claimed to be a test of maximum stringency, but as having a very wide field of application, especially when the constants of unknown parent populations are given their most probable values. The method which proceeds from these values seems to him as effective as attempting to find tests, which involve only sample values.

(iii) The P_{λ_n} test involves determining probability integrals, but tables of such integrals are now largely available and more will shortly be published.

(iv) It appeals first to the principle of independent probabilities, to ascertain the probability of more improbable *individual* occurrences, and then starting from this probability measures the probability of all sets of occurrences,—not necessarily greater in each individual variate but more improbable as a whole set.

(v) A number of illustrations are provided to indicate the breadth of the method, and in particular its value in the comparison of hypotheses.

(vi) The writer endeavours to emphasise a point which has, he thinks, not always been sufficiently regarded, namely that because a set of occurrences is found on a selected hypothesis H_1 not to be very improbable by test A, it does not follow that that hypothesis may be safely regarded as applying to the occurrences. A more stringent test B may show H_1 to be very improbable, or either or both tests, A and B, may show another hypothesis H_2 to be far more probable. In other words a test may suffice to allow us reasonably to reject a hypothesis, but only rarely (and generally in the cases where there is large previous experience) justifies us in accepting the hypothesis as a rule of conduct, or as a mode of extracting further information from our data.

(vii) Lastly, emphasis should be laid on the point that while probability integrals for a given investigation should all be measured in one direction, that direction may initially be *either* direction. In other words, a very high P_{λ_n} is calculated to arouse our suspicion as well as a very low P_{λ_n}. Really this warning needs to be borne in mind with nearly all tests, in particular with the P, χ^2 test. P_{χ^2} is a probability integral of the χ^2 curve measured in a particular direction, but there is no more valid reason for initially measuring it in one and not the opposite direction, than in the case of the normal curve probability integral, and, when the

size of the sample is not too small, χ^2 approaching zero, or P_{χ^2} approaching unity is as definite a warning as χ^2 very large and P_{χ^2} approaching zero. These considerations are equally valid when we consider P_{λ_n}, which should approach the value ·5 with increasing size of sample if the probability integrals are truly random, but marked deviation *either* way from this value is a warning that something is improbable either in the sampling or in the hypothesis from which the probability integrals have been deduced.

The present paper would have been impossible without the use of the *Incomplete* B- *and* Γ-*Function Tables*. The author has gratefully to acknowledge the aid in computing work of Miss F. N. David under a grant from the Department of Scientific and Industrial Research.

NOTE, *added December* 6, 1933.

After this paper had been set up Dr Egon S. Pearson drew my attention to Section 21·1 in the Fourth Edition of Professor R. A. Fisher's *Statistical Methods for Research Workers*, 1932. Professor Fisher is brief, but his method is essentially what I had thought to be novel. He uses, however, a χ^2 method, not my incomplete Γ-function solution; this explains the relation referred to in the footnote on p. 383 of my paper. As my paper was already set up and illustrates, more amply than Professor Fisher's two pages, some of the advantages and some of the difficulties of the new method, which may be helpful to students, I have allowed it to stand.

K. P.

MISCELLANEA.

(i) The Distribution of β_2 in samples of 4 from a Normal Universe.

By A. T. McKAY, M.Sc.

The estimated value of $\frac{1}{4}\beta_2$ is the statistic

$$\theta = \frac{\sum_1^4 (x_r - \bar{x})^4}{\left\{\sum_1^4 (x_r - \bar{x})^2\right\}^2} \quad \text{.....(1).}$$

Let us employ the orthogonal transformations

$$\left.\begin{aligned} 2x_1 &= \ \ y_1 + y_2 + y_3 + y_4 \\ 2x_2 &= -y_1 + y_2 - y_3 + y_4 \\ 2x_3 &= -y_1 - y_2 + y_3 + y_4 \\ 2x_4 &= \ \ y_1 - y_2 - y_3 + y_4 \end{aligned}\right\} \quad \text{.....(2),}$$

noting that $\sum_1^4 x_r^2 = \sum_1^4 y_r^2$, $\sum_1^4 (x_r - \bar{x})^2 = \sum_1^3 y_r^2$, $\bar{x} = \frac{1}{2}y_4$, $2(x_1 - \bar{x}) = \sum_1^3 y_r$, etc., then

$$16\sum_1^4 (x_r - \bar{x})^4 = \{(y_1+y_2+y_3)^4 + (y_1+y_3-y_2)^4 + (y_1+y_2-y_3)^4 + (y_2+y_3-y_1)^4\} \quad \text{.....(3).}$$

Since the expression on the right-hand side of (3) is cyclical and unaltered by a change of sign of any variable, we may infer that

$$16\sum_1^4 (x_r - \bar{x})^4 = A(y_1^2+y_2^2+y_3^2)^2 + B(y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2) \quad \text{.....(4).}$$

Giving suitable values to the variables, we can readily find $A=4$, $B=16$, whence

$$\theta_1 = (\theta - \tfrac{1}{4}) = \frac{y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2}{(y_1^2+y_2^2+y_3^2)^2} \quad \text{.....(5).}$$

Now since the expression on the right-hand side of (1) has four variables but only three degrees of freedom, we should be able to ascertain the range of fluctuation of β_2 by considering the expression (5). Write $y_1^2 = X$, $y_2^2 = Y$, $y_3^2 = Z$; then

$$\theta_1 = \frac{XY + YZ + ZX}{(X+Y+Z)^2} \quad \text{.....(6).}$$

Differentiating partially with respect to each of the three variables in turn and equating to zero, we derive

$$\left.\begin{aligned} Y(Y-X) + Z(Z-X) &= 0 \\ X(X-Y) + Z(Z-Y) &= 0 \\ X(X-Z) + Y(Y-Z) &= 0 \end{aligned}\right\} \quad \text{.....(7),}$$

from which it follows that there is a turning value when $X = Y = Z$. Further, since

$$\frac{\partial^2\theta_1}{\partial X^2} = \frac{\partial^2\theta_1}{\partial Y^2} = \frac{\partial^2\theta_1}{\partial Z^2} = \frac{-2}{27a^2} \quad \text{.....(8),}$$

when $X=Y=Z=a>0$, we can conclude that

$$\left.\begin{aligned} 0 \leqslant \beta_1 \leqslant \tfrac{1}{3} \\ 1 \leqslant \beta_2 \leqslant \tfrac{7}{3} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

irrespective of the character of the parent universe.

Let us now return to our main problem. We see from (5) that it is necessary to integrate

$$\frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}\sum_1^3 y_r^2} dy_1 dy_2 dy_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

over a triply infinite field conditioned by

$$w < \frac{y_1^2 y_2^2 + y_2^2 y_3^2 + y_3^2 y_1^2}{(y_1^2 + y_2^2 + y_3^2)^2} < w + \delta w \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Transform to polar coordinates by writing $y_1 = r\cos\theta$, $y_2 = r\sin\theta\,.\cos\phi$, $y_3 = r\sin\theta\,.\sin\phi$, hence equation (10) becomes

$$\frac{\sin\theta}{(2\pi)^{\frac{3}{2}}} d\theta\,.\,d\phi\,.\,r^2 e^{-r^2/2}\,.\,dr \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and condition (11) reduces to

$$w < \sin^2\theta\,.\cos^2\theta + \sin^4\theta\,.\sin^2\phi\,.\cos^2\phi < w + \delta w \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$.

Integrating out the r-term in (12) and writing $\cos\theta = -x$, $2\phi = \Phi$ in (12) and (13), we may derive $(2/\pi)\,dx\,.\,d\Phi$ in place of (12) and

$$w < x^2(1-x^2) + \tfrac{1}{4}(1-x^2)^2 \sin^2\Phi < w + \delta w \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

where now $0 \leqslant x \leqslant 1$ and $0 \leqslant \Phi \leqslant \pi/2$ indicate the limits of the field of integration which is to be conditioned by the last inequality. The moments of the distribution of w about the origin are thus given by

$$\nu_k = \frac{2}{\pi} \int_0^{\pi/2} \int_0^1 \{x^2(1-x^2) + \tfrac{1}{4}(1-x^2)^2 \sin^2\Phi\}^k \, dx\, d\Phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

By expanding the integrand by the binomial theorem and performing the integration term by term, we find

$$\nu_k = \frac{1}{2\Gamma\left(\frac{4k+3}{2}\right)} \left\{ \Gamma(k+1)\,\Gamma(k+\tfrac{1}{2}) + \frac{1}{2} \frac{\Gamma(k-\frac{1}{2})\,k\Gamma(k+2)}{4\,(1\,!)^2} \right.$$

$$\left. + \frac{1}{2} \cdot \frac{3}{2} \frac{\Gamma(k-\frac{3}{2})\,k(k-1)\,\Gamma(k+3)}{4^2\,(2\,!)^2} + \ldots \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Hence $\nu_1 = \frac{1}{5}$, $\nu_2 = \frac{1}{21}$, $\nu_3 = \frac{61}{5005}$, $\nu_4 = \frac{277}{85085}$, so that if μ refers to the moments of β_2 about the mean, $\mu_2 = 0{\cdot}12190$, $\mu_3 = -0{\cdot}02455$, $\mu_4 = 0{\cdot}034276$, $B_1 = 0{\cdot}33273$, $B_2 = 2{\cdot}30645$.

Returning now to (15) and the remarks which precede it, we conclude that if $\phi(w)$ is the distribution which is being sought, then

$$\phi(w) = \frac{2}{\pi} \int \frac{\partial\Phi}{\partial w}\, dx = \frac{1}{\pi} \int \frac{dx}{\{(w - x^2 + x^4)\,(\frac{1}{4}(1-x^2)^2 - w + x^2 - x^4)\}^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

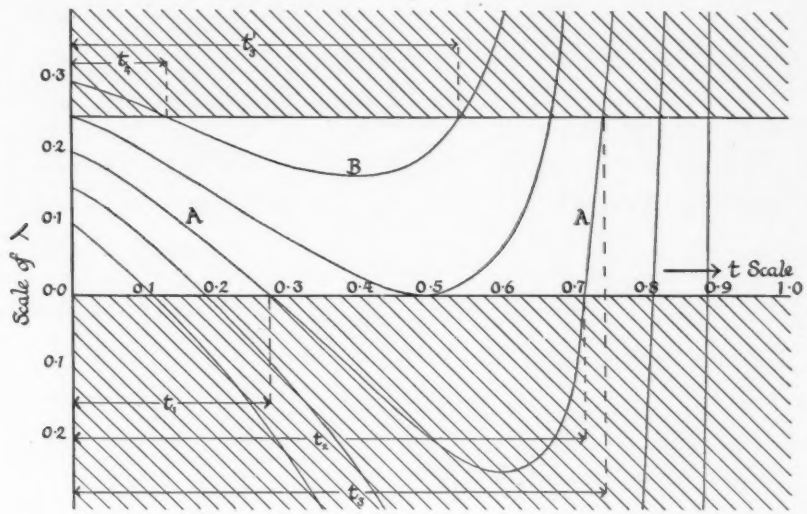
where the limits of the integral are such that x runs from 0 to 1 subject to the condition

$$0 \leqslant 4\frac{\{w - x^2 + x^4\}}{(1-x^2)^2} \leqslant 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

In order to find the appropriate limits, we consider the family of curves defined by

$$\lambda = \frac{w - t + t^2}{(1-t)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

and shown in the following diagram. We note from Equation (18) that only values of λ in the unshaded part of the diagram are consistent with our requirements.



There are plainly two cases to consider.

Case (i). $0 \leqslant w \leqslant \frac{1}{4}$.

Let the curve marked A be regarded as a typical curve for this case. Then there are two independent ranges which satisfy our conditions, i.e.

$$(1)\ 0 \leqslant t \leqslant t_1,$$

and

$$(2)\ t_2 \leqslant t \leqslant t_3.$$

Case (ii). $\frac{1}{4} \leqslant w \leqslant \frac{1}{3}$.

Here we take B as a typical curve, which shows that there is only one possible range, viz.

$$(3)\ t_4 \leqslant t \leqslant t_3'.$$

By writing $\lambda = 0$ and $\lambda = \frac{1}{4}$ in Equation (19), we readily find

$$\left.\begin{aligned} t_1 &= \tfrac{1}{2}(1 - \sqrt{1-4w}) \\ t_2 &= \tfrac{1}{2}(1 + \sqrt{1-4w}) \\ t_3 = t_3' &= \tfrac{1}{3}(1 + 2\sqrt{1-3w}) \\ t_4 &= \tfrac{1}{3}(1 - 2\sqrt{1-3w}) \end{aligned}\right\} \quad \text{.....(20).}$$

If now in (17) we change the variable by writing $x^2 = t$, we conclude

$$\phi(w) = \frac{1}{2\pi}\left(\int_0^{t_1} + \int_{t_2}^{t_3}\right) F(t)\,dt \qquad 0 \leqslant w \leqslant \tfrac{1}{4} \quad \text{.....(21)}$$

$$= \frac{1}{2\pi}\int_{t_4}^{t_3} F(t)\,dt \qquad \tfrac{1}{4} \leqslant w \leqslant \tfrac{1}{3} \quad \text{.....(22),}$$

where

$$F(t) = \{t(w - t + t^2)(\tfrac{1}{4}(1-t)^2 - w + t - t^2)\}^{-\frac{1}{2}} \quad \text{.....(23).}$$

Let us now make the substitution

$$t=\tfrac{1}{3}\{1+2\sqrt{1-3w}\sin\theta\} \quad \text{.....(24),}$$

then

$$F(t)\frac{dt}{d\theta}=6\{(9w-2)-2(1-3w)^{\frac{3}{2}}\sin 3\theta\}^{-\frac{1}{2}} \quad \text{.....(25).}$$

Hence writing $\frac{1}{2}(1-3w)^{-\frac{1}{2}}=\sin\alpha$ when $0\leqslant w\leqslant\frac{1}{4}$, we find

$$\phi(w)=\frac{1}{2\pi}\left(\int_{-\alpha}^{\alpha-\pi/3}+\int_{2\pi/3-\alpha}^{\pi/2}\right)F(t)\frac{dt}{d\theta}\,d\theta \qquad \text{when } 0\leqslant w\leqslant\tfrac{1}{4}\text{.....(26)}$$

$$=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}F(t)\frac{dt}{d\theta}\,d\theta \qquad \text{when } \tfrac{1}{4}\leqslant w\leqslant\tfrac{1}{3}\text{.....(27).}$$

Substituting $\theta=\frac{1}{3}\xi-\pi/6$, $3\alpha=\pi/2+\beta$, and noting that $\sin 3\alpha=\frac{1}{2}(2-9w)/(1-3w)^{\frac{3}{2}}$, we deduce

$$\phi(w)=\frac{3}{\pi(1-3w)^{\frac{1}{2}}}\int_0^{\beta}\frac{d\xi}{(2\cos\xi-2\cos\beta)^{\frac{1}{2}}} \qquad 0\leqslant w\leqslant\tfrac{1}{4}\text{.....(28)}$$

$$=\frac{1}{\pi}\int_0^{3\pi}\frac{d\xi}{\{(9w-2)+2(1-3w)^{\frac{3}{2}}\cos\xi\}} \qquad \tfrac{1}{4}\leqslant w\leqslant\tfrac{1}{3}\text{.....(29).}$$

The first of these integrals (28) is a Mehler Integral, hence

$$\phi(w)=\frac{3}{2(1-3w)^{\frac{1}{2}}}\left\{P_{-\frac{1}{2}}(\cos\beta)=P_{-\frac{1}{2}}(-\gamma)=F\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{1+\gamma}{2}\right)\right\} \quad \text{.....(30)}$$

when $0\leqslant w\leqslant\frac{1}{4}$ and $\gamma=(9w-2)/2(1-3w)^{\frac{3}{2}}$.

The second integral (29) may be written

$$\phi(w)=\frac{1}{\pi(9w-2)^{\frac{1}{2}}}\int_0^{3\pi}\frac{d\xi}{(1+(\cos\xi)/\gamma)^{\frac{1}{2}}} \quad \text{.....(31),}$$

the integrand of which can be expanded by the binomial theorem and integrated term by term to yield

$$\phi(w)=\frac{3}{(9w-2)^{\frac{1}{2}}}F\left(\tfrac{1}{4},\tfrac{3}{4};1;\frac{1}{\gamma^2}\right) \qquad \text{when } \tfrac{1}{4}\leqslant w\leqslant\tfrac{1}{3} \quad \text{.....(32).}$$

We note in passing that this last equation may also be written

$$\phi(w)=\frac{3}{\pi(1-3w)^{\frac{1}{2}}}Q_{-\frac{1}{2}}(-\gamma) \quad \text{.....(33),}$$

where Q represents the Legendre Function of the second kind.

By writing $w=\frac{1}{4}(x-1)$ in equations (30) and (32), we finally find that the distribution of β_2 in samples of four from a normal universe is given by $f(x)$, where

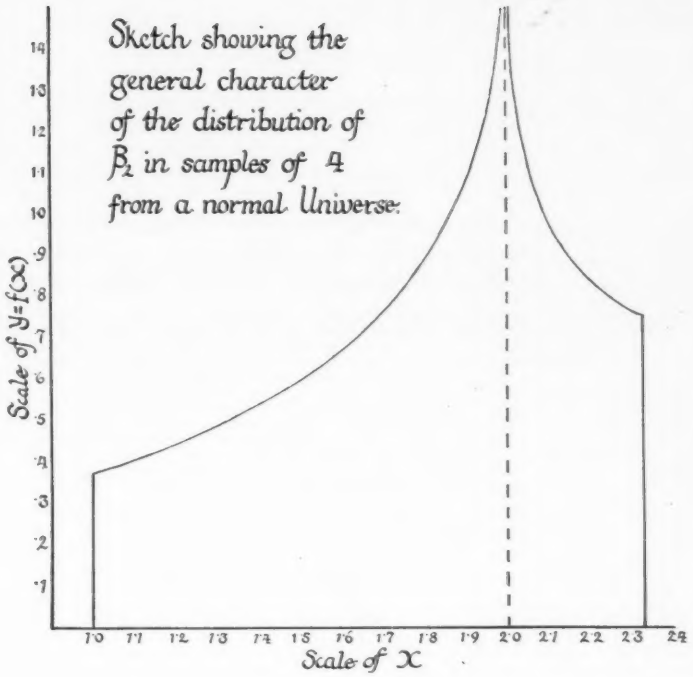
$$f(x)=\frac{3}{2^{\frac{1}{2}}(7-3x)^{\frac{1}{2}}}F\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{1+g}{2}\right) \qquad \text{when } 1\leqslant x\leqslant 2\text{.....(34)}$$

$$=\frac{3}{2(9x-17)^{\frac{1}{2}}}F\left(\tfrac{1}{4},\tfrac{3}{4};1;\frac{1}{g^2}\right) \qquad \text{when } 2\leqslant x\leqslant\tfrac{7}{3}\text{.....(35),}$$

where $g=(9x-17)/(7-3x)^{\frac{3}{2}}$.

The chief characteristics of this distribution are shown in the accompanying diagram. Owing to the extreme difficulty of calculating the values of the hyper-geometric functions this had to be omitted, so that the two curves have merely been sketched in. Should, however, it prove desirable to have an exact knowledge of the form of these curves, the procedure indicated in the previous paper* would most likely prove of considerable assistance.

* *Biometrika*, Vol. XXV. Parts I and II (1933) (Miscellanea).


Sketch showing the general character of the distribution of β_2 in samples of 4 from a normal Universe.
Scale of $y=f(x)$
Scale of x

(ii) A Note on the Distribution of Range in Samples of n.

By A. T. McKAY, M.Sc., and E. S. PEARSON, D.Sc.

In a recent paper* one of the present writers has provided a table giving certain percentage limits for the distribution of range in samples from a normal population. These limits were obtained on the assumption that the distribution could be adequately represented by Pearson curves having the appropriate moment coefficients. The theoretical treatment in Section (1) below, while following the general method of approach previously employed, leads to certain new results regarding the form of the range curve at the terminals, and also provides the exact distribution of range in the case of samples of 3 from a normal population. In this latter case, therefore, it makes possible a check on the accuracy of the published table.

(1) *Theoretical Treatment.*

Let $x_1, x_2, \ldots, x_n$ be a random sample of n from a universe defined in the interval $(-b, a)$ for x by the probability function

$$y=f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

* E. S. Pearson, *Biometrika*, Vol. XXIV. p. 416.

To find the distribution of the range, i.e. the numerical value of the difference between the greatest and least observations in a random sample of n, we require to integrate

$$f(x_1)f(x_2)\dots f(x_n)\,dx_1\,dx_2\dots dx_n \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

over an appropriate field.

Consider any pair of values x_1 and x_2 selected from the group of n. Then the compound probability that these two values are the extremes of the group, x_1 being the least and x_2 the greatest, and at the same time give a value of the range lying between w and $w+\delta w$, is found by integrating the expression (2) over all possible values of x_1 and x_2, subject to the conditions

$$\left.\begin{aligned} &x_1<x_r<x_2 \qquad\qquad r=3,\ 4,\ \dots\dots\ n\\ \text{and}\quad &w<|x_2-x_1|<w+\delta w \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots(3).$$

Since, however, the pair of values x_1 and x_2 can be selected in $n(n-1)$ different ways, the total probability required is to be derived by multiplying the results of the integration by $n(n-1)$.

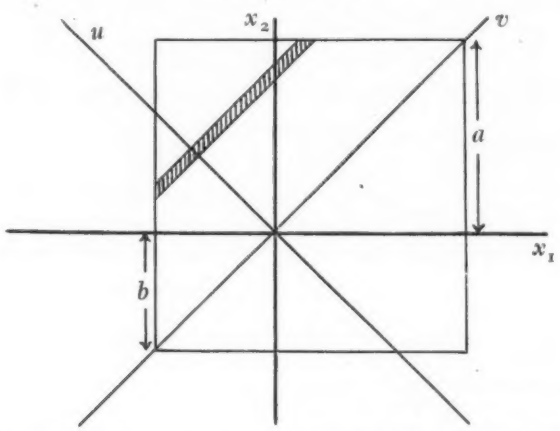
Thus if the required distribution function is $\phi(w)$, we find

$$\phi(w)\,\delta w = n(n-1)\iint f(x_1)f(x_2)\left(\int_{x_1}^{x_2} f(x)\,dx\right)^{n-2} dx_1\,dx_2 \quad \dots\dots\dots\dots\dots\dots(4),$$

where the limits of the integrals for x_1 and x_2 are chosen to satisfy the second condition of Equation (3).

Let us now transform the variables by the substitutions

$$(x_2-x_1)=u\sqrt{2},$$
$$(x_2+x_1)=v\sqrt{2}.$$



Then condition (3) requires that we integrate throughout the shaded strip in the diagram. The limits for u are therefore u and $u+\delta u$, while those for v are $(u-b\sqrt{2})$ and $(a\sqrt{2}-u)$. Hence,

$$\phi(w)\,dw = n(n-1)\int_{u-b\sqrt{2}}^{a\sqrt{2}-u}\int_{u}^{u+\delta u} f\left(\frac{u+v}{\sqrt{2}}\right)f\left(\frac{v-u}{\sqrt{2}}\right)\left(\int_{(v-u)/\sqrt{2}}^{(v+u)/\sqrt{2}} f(x)\,dx\right)^{n-2} du\,dv \dots\dots(5),$$

or substituting $u=w/\sqrt{2}$ and $v=\sqrt{2}t$,

$$\phi(w)=n(n-1)\int_{\frac{1}{2}w-b}^{a-\frac{1}{2}w} f(t+\tfrac{1}{2}w)f(t-\tfrac{1}{2}w)\left(\int_{t-\frac{1}{2}w}^{t+\frac{1}{2}w} f(x)\,dx\right)^{n-2} dt \quad \dots\dots\dots\dots\dots(6),$$

where the range for w is from 0 to $(a+b)$.

Example 1. Distribution of the Range in samples of n from a Rectangular Universe*.

In this case $f(x)=\frac{1}{2}$, $a=b=1$, whence

$$\phi(w)=\frac{n(n-1)}{2^n}w^{n-2}(2-w) \quad \text{.....(7).}$$

Example 2. Distribution of the Range in samples of n from a Straight Line Universe.

Take $y=f(x)=2(1-x) \qquad 0\leqslant x\leqslant 1,$

$=0 \qquad\qquad x<0 \text{ and } >1,$

so that $b=0$ and $a=1$, then after a simple integration and reduction we find

$$\phi(w)=\frac{n(n-1)}{2}w^{n-2}\left\{\frac{2w^{n+1}}{(n^2-1)}+\frac{(2-w)^{n+1}}{(n+1)}-\frac{w^2(2-w)^{n-1}}{(n-1)}\right\} \quad \text{.....(8).}$$

Example 3. The distribution of the Range in samples of 3 from a Normal Universe.

Write $n=3$, $f(x)=\dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$ and $a=b=\infty$ in Equation (6), whence

$$\phi(w)=\frac{6e^{-w^2/4}}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}e^{-t^2}\int_{t-w/2}^{t+w/2}e^{-x^2/2}\,dx\,dt \quad \text{.....(9).}$$

Putting $x=(y+t)$ and changing the order of integration, we get

$$\phi(w)=\frac{6e^{-w^2/4}}{(2\pi)^{3/2}}\int_{-w/2}^{w/2}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(3t^2+2yt+y^2)}\,dt\,dy \quad \text{.....(10)}$$

$$=\frac{6e^{-w^2/4}}{\pi\sqrt{3}}\int_{0}^{w/2}e^{-y^2/3}\,dy$$

$$=\frac{6e^{-w^2/4}}{\pi\sqrt{2}}\int_{0}^{w/\sqrt{6}}e^{-u^2/2}\,du \qquad (\text{writing } u=\sqrt{\tfrac{2}{3}}y) \quad \text{.....(11).}$$

The kth moment, μ_k, of this distribution about the origin is

$$\mu_k=\frac{6}{\pi\sqrt{2}}\int_0^{\infty}w^k e^{-w^2/4}\int_0^{w/\sqrt{6}}e^{-u^2/2}\,du\,dw=\frac{6}{\pi\sqrt{2}}\int_0^{\infty}\int_0^{1/\sqrt{6}}w^{k+1}e^{-(1+2v^2)w^2/4}\,dv\,dw \quad \text{.....(12).}$$

Writing $v=u/w$, changing the order of integration and integrating for w in the last equation, we find

$$\mu_k=\frac{2^{k+1}\times 6\,.\,\Gamma\left(\frac{k}{2}+1\right)}{\pi\sqrt{2}}\int_0^{1/\sqrt{6}}\frac{dv}{(1+2v^2)^{\frac{1}{2}(k+2)}} \quad \text{.....(13).}$$

Substituting $v=(1/\sqrt{2})\tan\theta$ in the latter integral, we finally derive

$$\mu_k=\frac{3}{\pi}\times 2^{k+1}\,\Gamma\left(\frac{k}{2}+1\right)\int_0^{\pi/6}\cos^k\theta\,d\theta \quad \text{.....(14).}$$

To obtain some idea of the character of the range curve in the neighbourhood of its terminals, for the case in which the parent universe extends from $-\infty$ to $+\infty$, we can approximate as follows.

Supposing w is very large, the term

$$\left(\int_{t-w/2}^{t+w/2}f(x)\,dx\right)^{n-2}$$

* This has already been given by J. Neyman and E. S. Pearson, *Biometrika*, Vol. xx[A]. p. 210.

of Equation (6) tends to unity, thus we infer that

$$\phi(w)=n(n-1)\int_{-\infty}^{\infty} f\left(t+\frac{w}{2}\right) f\left(t-\frac{w}{2}\right) dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

provided w is large enough. On the other hand, when w is very small, we may write

$$\int_{t-w/2}^{t+w/2} f(x)\,dx = w f(t),$$

whence we infer that

$$\phi(w)=n(n-1)\,w^{n-2}\int_{-\infty}^{\infty} f\left(t+\frac{w}{2}\right) f\left(t-\frac{w}{2}\right) [f(t)]^{n-2}\,dt \quad \ldots\ldots\ldots\ldots\ldots (16),$$

provided w is small enough.

There does not appear to be any ready means of determining the accuracy of the approximations (15) and (16), but it is possible that they may be of assistance in selecting a suitable approximation curve, when proceeding to find a range distribution by indirect methods.

(2) *Application in the Case of Samples of* 3 *from a Normal Population.*

Using Equation (11) the ordinates of the frequency curve, $\phi(w)$, were obtained with the help of tables of the normal curve. These ordinates have been compared in Table I with those previously calculated from a Pearson Type I curve*. It will be seen that the greatest relative difference occurs at the start of the curves. From the practical point of view, we are concerned

TABLE I.

Comparison of ordinates of {(A) *true theoretical range curve* ($n=3$). (B) *fitted Pearson curve.*

Range	Ordinates		Range	Ordinates		Range	Ordinates	
	A.	B.		A.	B.		A.	B.
·1	5·5	3·6	1·7	42·1	41·2	3·3	9·1	9·6
·2	10·9	9·2	1·8	40·5	39·4	3·4	7·8	8·3
·3	16·1	15·2	1·9	38·6	37·5	3·5	6·7	7·1
·4	21·1	21·1	2·0	36·5	35·3	3·6	5·7	6·1
·5	25·7	26·5	2·1	34·2	33·1	3·7	4·8	5·2
·6	29·9	31·3	2·2	31·8	30·9	3·8	4·0	4·3
·7	33·7	35·4	2·3	29·4	28·6	3·9	3·4	3·6
·8	36·9	38·7	2·4	27·0	26·3	4·0	2·8	3·0
·9	39·6	41·4	2·5	24·6	24·1	4·1	2·3	2·5
1·0	41·8	43·3	2·6	22·2	21·9	4·2	1·9	2·0
1·1	43·3	44·5	2·7	20·0	19·8	4·3	1·5	1·6
1·2	44·4	45·1	2·8	17·8	17·8	4·4	1·2	1·3
1·3	44·9	45·2	2·9	15·8	15·9	4·5	1·0	1·0
1·4	44·8	44·8	3·0	13·9	14·1	4·6	0·8	0·8
1·5	44·3	43·9	3·1	12·2	12·5	4·7	0·6	0·6
1·6	43·4	42·7	3·2	10·6	11·0	4·8	0·5	0·5

N.B.—The unit for range is the population standard deviation, and the curves are calculated so that the area under each is 100. The tails of the curves extend of course beyond $w=4·8$.

* *Loc. cit.* This curve was made to start at $w=0$, and given the correct first three moment coefficients.

with the extent of error in position of the percentage limits calculated from the approximate curve. This may be seen in Table II, where the limits are given for:

A. The true curve of Equation (11), obtained by quadrature and backward interpolation.

B. The Pearson curve; the limits are taken from the row, $n=3$, of the published table*.

C. A Normal curve having correct mean and standard deviation, namely $w=1{\cdot}6926$, $\sigma=0{\cdot}8884$.

TABLE II.

Percentage limits for range ($n=3$) calculated by various methods.

Curve used	Lower Limits				Upper Limits			
	0·5 %	1 %	5 %	10 %	10 %	5 %	1 %	0·5 %
A. True curve	·13	·19	·43	·62	2·90	3·31	4·12	4·43
B. Pearson curve	·17	·22	·45	·63	2·92	3·34	4·10	4·36
C. Normal curve	−·60	−·37	·23	·55	2·83	3·15	3·76	3·98

The largest difference between A and B (of ·07 %) occurs at the upper 0·5 % limit, the second largest (of ·04) at the lower 0·5 % limit; otherwise the differences are ·03 or less. Since the approximate method might be expected to give least satisfactory agreement for this case of $n=3$ (where the distribution curve for range has greatest skewness), these results seem to confirm the opinion previously given when publishing the tables: "the addition of a 3rd decimal place in the limits would clearly be meaningless, but the retention of the 2nd decimal appears worth while†."

Limits calculated by using a normal curve (C) have been shown in Table II to emphasise the fact that while the Pearson curve (B) may not lead to mathematically exact limits, it provides a far more accurate and useful approximation than can be obtained by the crude method (C).

(3) *Limiting Form of $\phi(w)$.*

Finally it seemed of interest to make a trial of the terminal formula (15). When the population sampled is normal, we have

$$\phi(w)=\frac{n(n-1)}{2\pi}\int_{-\infty}^{+\infty}e^{-(w^2/4+t^2)}\,dt=\frac{n(n-1)}{2\sqrt{\pi}}\,e^{-w^2/4} \quad \text{.....(17).}$$

Whence we obtain

$$A_w=\int_w^\infty \phi(w)\,dw=n(n-1)\int_x^{+\infty}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx=n(n-1)\times\tfrac{1}{2}(1-a_x) \quad \text{.....(18),}$$

where $x=\dfrac{1}{\sqrt{2}}w$, and a_x is in the notation of Sheppard's Tables.

* *Loc. cit.* p. 416. † *Loc. cit.* p. 405.

For $n=3$, we obtain from (18),

A_w	·05 (5 °/₀ limit)	·01 (1 °/₀)	·005 (0·5 °/₀)
w	3·39	4·15	4·46

the last two of which are in very close agreement with the true limits given in Table II.

For $n=50$ and $n=100$, the following comparisons are obtained:

	A_w	·05 (5 °/₀)	·01 (1 °/₀)	·005 (0·5 °/₀)
$n=50$	Equation (18)	5·80	6·31	6·52
	Table A of previous paper	5·64	6·23	6·45
$n=100$	Equation (18)	6·25	6·72	6·92
	Table A of previous paper	6·08	6·63	6·85

In these cases the true position of the limits is not of course known. It seems, however, likely that for $A_w \leqslant ·01$, the equation (18) may lead to an approximation of considerable practical value when limits are required beyond the range of those tabled, e.g. either for $A_w < ·005$, or $n>100$.

(iii) On a Recurrence Relation connected with the Double Bessel Functions $\mathcal{K}_{\tau_1,\tau_2}(x)$ and $T_{\tau_1,\tau_2}(x)$.

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A recurrence relation is required for

$$I_{\tau_1,\tau_2}=\int_0^x e^{-\rho x}\,T_{\tau_1,\tau_2}(x)\,dx,$$

where
$$T_{\tau_1,\tau_2}(x)=\frac{1}{\sqrt{\pi}}\,\frac{1}{2^\nu}\,\frac{1}{\Gamma(\nu+\frac{1}{2})}\,x^\nu \mathcal{K}_{\tau_1,\tau_2}(x) \quad \text{.....(xxxvii),}$$

and
$$\mathcal{K}_{\tau_1,\tau_2}(x)=\frac{\sqrt{\pi}}{2^\nu\,\Gamma(\nu+\frac{1}{2})}\,x^\nu\int_1^\infty e^{-xt}(t-1)^{\tau_1}\,(t+1)^{\tau_2}dt \quad \text{.....(xxix),}$$

where
$$\nu=\tfrac{1}{2}(\tau_1+\tau_2+1),$$

the numbers (xxxvii) and (xxix) referring to the definition of $T_{\tau_1,\tau_2}(x)$ and $\mathcal{K}_{\tau_1,\tau_2}(x)$ in Professor Karl Pearson's paper in *Biometrika*, Vol. xxv. pp. 158—178.

We have by (xliii) of the same paper

$$\frac{d}{dx}\left(T_{\tau_1,\tau_2}(x)\right)=\frac{-x}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)T_{\tau_1-1,\tau_2-1}(x)-\frac{\tau_1-\tau_2}{2\nu-1}\,T_{\tau_1,\tau_2}(x).$$

Also
$$e^{-\rho x}\frac{d}{dx}\,T_{\tau_1,\tau_2}(x)=\frac{d}{dx}\left(e^{-\rho x}T_{\tau_1,\tau_2}(x)\right)+\rho e^{-\rho x}T_{\tau_1,\tau_2}(x).$$

Write
$$e^{-\rho x}T_{\tau_1,\tau_2}(x)=\tilde{T}_{\tau_1,\tau_2}(x),$$

then
$$\frac{d}{dx}\left(\tilde{T}_{\tau_1,\tau_2}(x)\right)=-\frac{x}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)\tilde{T}_{\tau_1-1,\tau_2-1}(x)-\left(\rho+\frac{\tau_1-\tau_2}{2\nu-1}\right)\tilde{T}_{\tau_1,\tau_2}(x).$$

Integrating,

$$\tilde{T}_{\tau_1,\tau_2}(x)-\tilde{T}_{\tau_1,\tau_2}(0)=-\frac{1}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)\int_0^x x\,\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$
$$-\left(\rho+\frac{\tau_1-\tau_2}{2\nu-1}\right)I_{\tau_1,\tau_2} \quad \text{.....(1).}$$

Multiplying equation (xliii *bis*) by $e^{-\rho x}$ we have

$$\tilde{T}_{\tau_1,\tau_2}(x)=\left\{\frac{2\nu-2}{2\nu-1}+\frac{\tau_1-\tau_2}{(2\nu-1)^2}x\right\}\tilde{T}_{\tau_1-1,\tau_2-1}(x)-\frac{x}{2\nu-1}e^{-\rho x}\frac{d}{dx}T_{\tau_1-1,\tau_2-1}(x)$$

$$=\left\{\frac{2\nu-2}{2\nu-1}+\left(\frac{\tau_1-\tau_2}{(2\nu-1)^2}-\frac{\rho}{2\nu-1}\right)x\right\}\tilde{T}_{\tau_1-1,\tau_2-1}(x)-\frac{x}{2\nu-1}\frac{d}{dx}\{\tilde{T}_{\tau_1-1,\tau_2-1}(x)\}.$$

Integrating

$$I_{\tau_1,\tau_2}=\frac{2\nu-2}{2\nu-1}I_{\tau_1-1,\tau_2-1}+\left(\frac{\tau_1-\tau_2}{(2\nu-1)^2}-\frac{\rho}{2\nu-1}\right)\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$-\frac{1}{2\nu-1}\int_0^x x\frac{d}{dx}\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$=\frac{2\nu-2}{2\nu-1}I_{\tau_1-1,\tau_2-1}+\left(\frac{\tau_1-\tau_2}{(2\nu-1)^2}-\frac{\rho}{2\nu-1}\right)\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$-\frac{1}{2\nu-1}x\tilde{T}_{\tau_1-1,\tau_2-1}(x)+\frac{1}{2\nu-1}I_{\tau_1-1,\tau_2-1}$$

$$=I_{\tau_1-1,\tau_2-1}+\left(\frac{\tau_1-\tau_2}{(2\nu-1)^2}-\frac{\rho}{2\nu-1}\right)\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$-\frac{1}{2\nu-1}x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Using (1) to eliminate $\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$ we have

$$I_{\tau_1,\tau_2}=I_{\tau_1-1,\tau_2-1}+\frac{\frac{1}{2\nu-1}\left(\frac{\tau_1-\tau_2}{2\nu-1}-\rho\right)}{-\frac{1}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)}\left\{\tilde{T}_{\tau_1,\tau_2}(x)-\tilde{T}_{\tau_1,\tau_2}(0)+\left(\rho+\frac{\tau_1-\tau_2}{2\nu-1}\right)I_{\tau_1,\tau_2}\right\}$$

$$-\frac{1}{2\nu-1}x\tilde{T}_{\tau_1,\tau_2}(x),$$

$$I_{\tau_1,\tau_2}=I_{\tau_1-1,\tau_2-1}+\frac{\rho-\frac{\tau_1-\tau_2}{2\nu-1}}{1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2}\{\tilde{T}_{\tau_1,\tau_2}(x)-\tilde{T}_{\tau_1,\tau_2}(0)\}+\frac{\rho^2-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2}{1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2}I_{\tau_1,\tau_2}$$

$$-\frac{x}{2\nu-1}\tilde{T}_{\tau_1,\tau_2}(x).$$

Writing $$\frac{\tau_1-\tau_2}{\tau_1+\tau_2}=\frac{\tau_1-\tau_2}{2\nu-1}=\kappa,$$

$$\frac{1-\rho^2}{1-\kappa^2}I_{\tau_1,\tau_2}=I_{\tau_1-1,\tau_2-1}+\frac{\rho-\kappa}{1-\kappa^2}\{\tilde{T}_{\tau_1,\tau_2}(x)-\tilde{T}_{\tau_1,\tau_2}(0)\}-\frac{x}{2\nu-1}\tilde{T}_{\tau_1,\tau_2}(x),$$

i.e. $$I_{\tau_1,\tau_2}=\frac{1-\kappa^2}{1-\rho^2}I_{\tau_1-1,\tau_2-1}+\frac{\rho-\kappa}{1-\rho^2}\{\tilde{T}_{\tau_1,\tau_2}(x)-\tilde{T}_{\tau_1,\tau_2}(0)\}-\frac{1-\kappa^2}{1-\rho^2}\frac{x}{2\nu-1}\tilde{T}_{\tau_1,\tau_2}(x),$$

which is the required recurrence formula.

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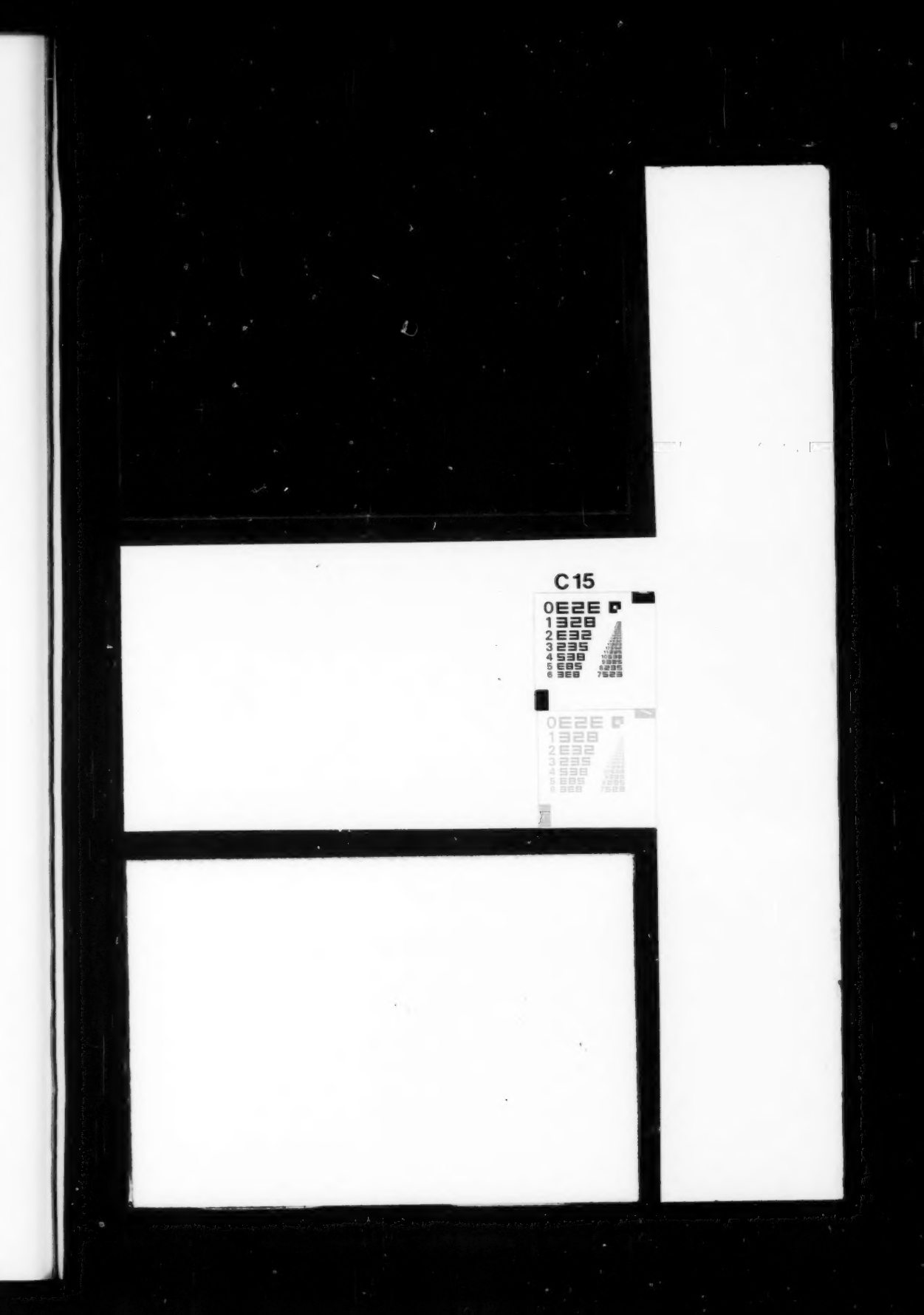
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